Parameter Redundancy and Identifiability in Ecological Models

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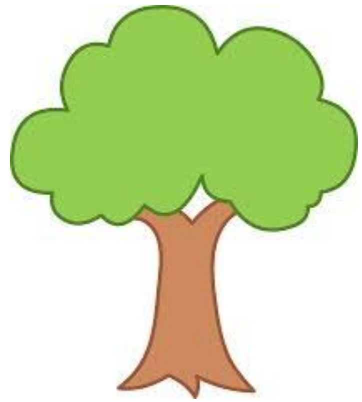
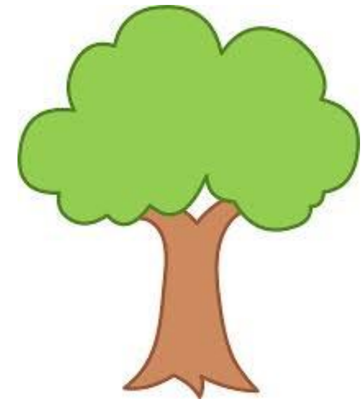
Introduction

Occupancy Model example





Species present and detected

Prob $= \psi p$

Species present but not detected

Prob $= \psi(1-p)$

Species absent

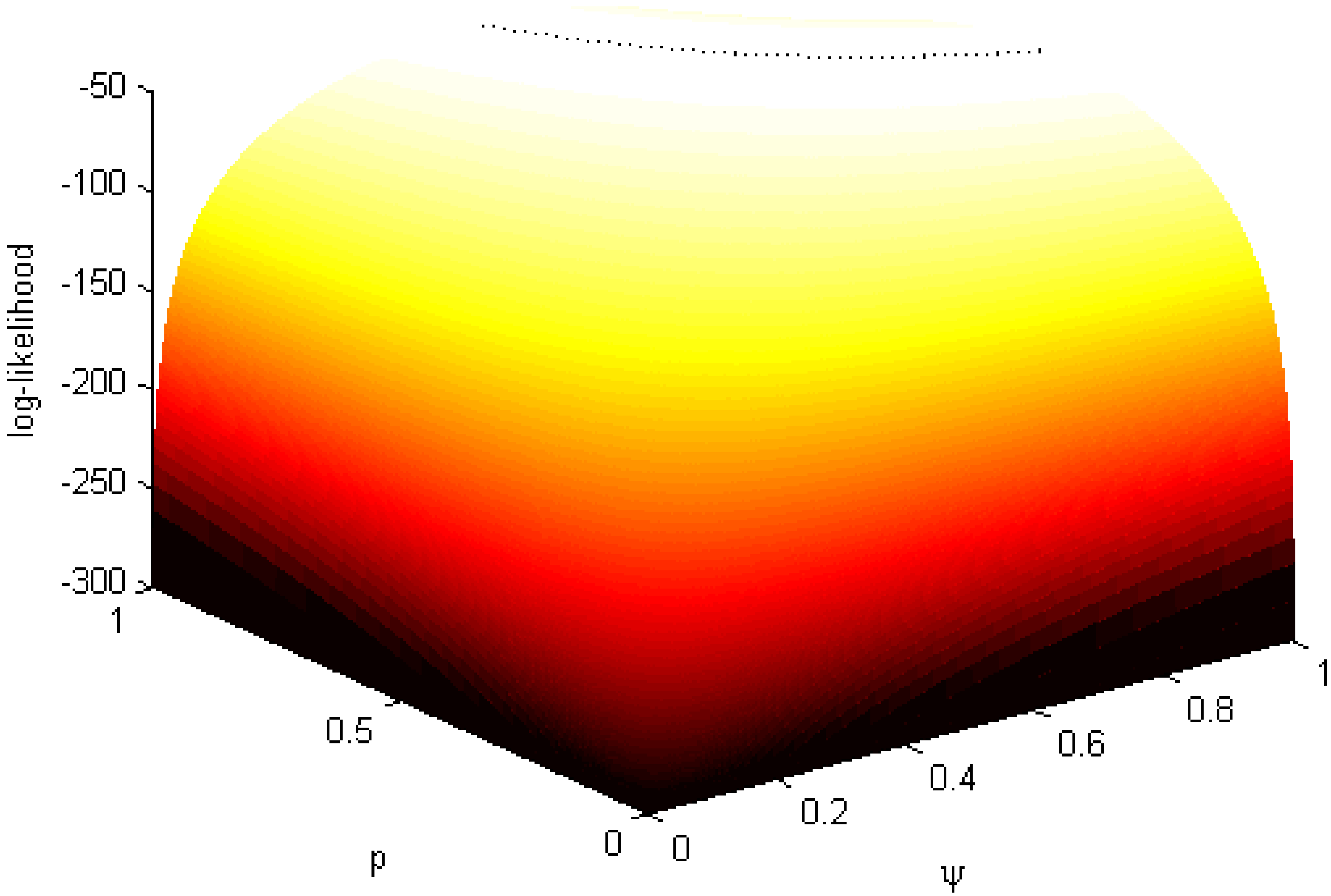
Prob $= 1-\psi$

Prob seen $= \psi p$

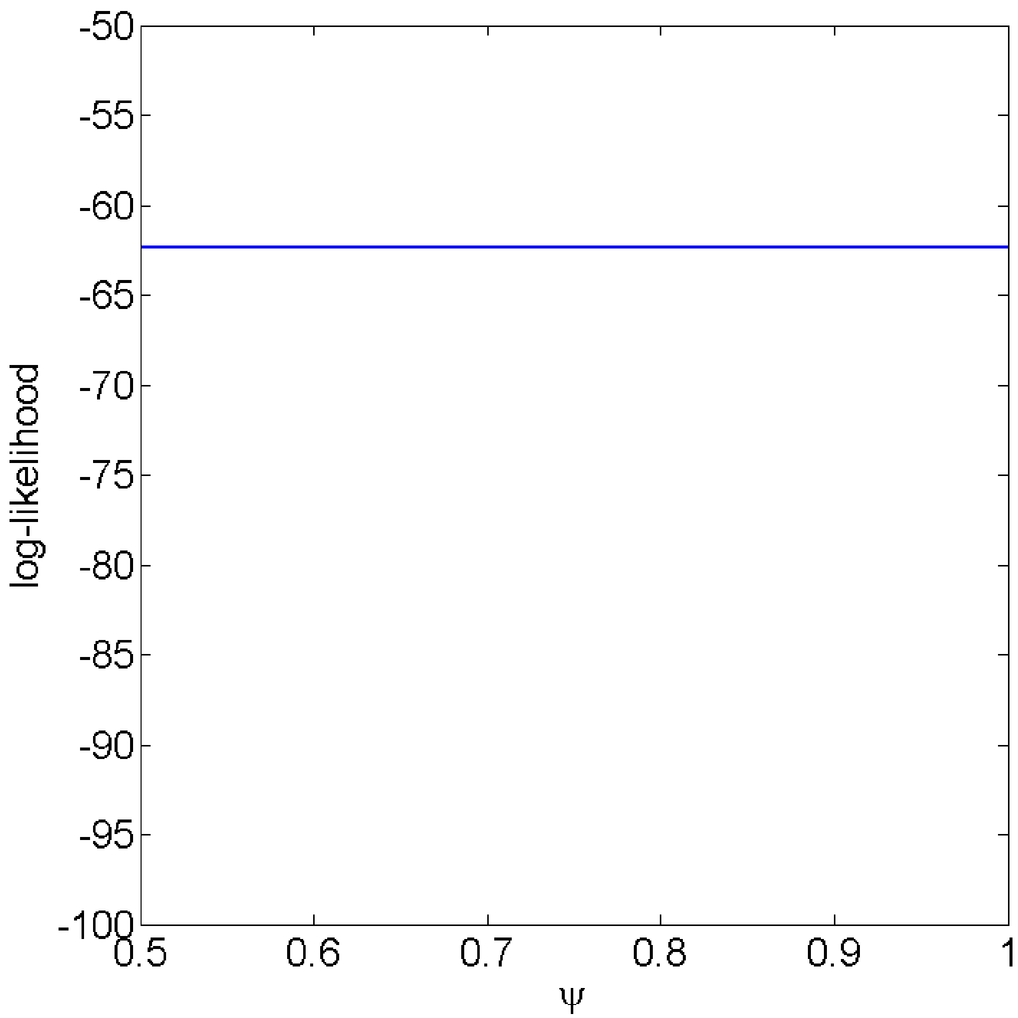
Prob not seen $= \psi(1-p) + 1 - \psi$
$= 1 - \psi p$

- Parameters: ψ $-$ site is occupied, p – species is detected.
- Can only estimate ψp rather than ψ and p.
- Model is parameter redundant or parameters are non-identifiable.
- (Solution: MacKenzie et al, 2002, robust design; visit sites several times in one season).

Occupancy Model - Likelihood Surface



Occupancy Model - Likelihood Profile



Flat profile = Parameter Redundant

Parameter Redundancy and Identifiability

- Suppose we have a model $M(\theta)$ with parameters θ. A model is globally (locally) identifiable if $M(\theta_1) = M(\theta_2)$ implies that $\theta_1 = \theta_2$ (for a neighbourhood of θ).
- A model is parameter redundant if it can be reparameterised in terms of a smaller set of parameters. A parameter redundant model is non-identifiable.
- There are several different methods for detecting parameter redundancy, including:
  - numerical methods (e.g. Viallefont et al, 1998)
  - symbolic methods (e.g. Cole et al, 2010)
  - hybrid symbolic-numeric method (Choquet and Cole, 2012).
- Generally involves calculating the rank of a matrix, which gives the number of parameters that can be estimated.

Problems with Parameter Redundancy

- There will be a flat ridge in the likelihood of a parameter redundant model (Catchpole and Morgan, 1997), resulting in more than one set of maximum likelihood estimates.
- Numerical methods to find the MLE will not pick up the flat ridge, although it could be picked up by trying multiple starting values and looking at profile log-likelihoods. If a parameter redundant model is fitted will get biased parameter estimates.
- The Fisher information matrix will be singular (Rothenberg, 1971) and therefore the standard errors will be undefined.
- However the exact Fisher information matrix is rarely known. Standard errors are typically approximated using a Hessian matrix obtained numerically. Can get explicit (wrong) estimates of standard errors in some cases.
- Model selection is based on identifiable models (e.g. AIC needs number of parameters can estimate).
- Recommended you check the identifiability of a model before fitting (or at least as part of model fitting process).

Symbolic Method (Occupancy Example)

- Catchpole and Morgan (1997), Catchpole et al (1998), Cole et al (2010)
- Exhaustive summary: $\boldsymbol{\kappa} = \begin{bmatrix} \psi p \\ 1 - \psi p \end{bmatrix}$
- Parameters: $\boldsymbol{\theta} = [\psi \;\; p]$
- Derivative matrix: $\boldsymbol{D} = \frac{\partial \boldsymbol{\kappa}}{\partial \boldsymbol{\theta}} = \begin{bmatrix} p & -p \\ \psi & -\psi \end{bmatrix}$
- Rank of ***D***, *r*, is number of estimable parameters.
- If *r* is less than number of parameters, *q*, the model is parameter redundant. Rank is 1, model parameter redundant.
- Can find if any of original parameters are individually identifiable by solving $\boldsymbol{\alpha}^T \boldsymbol{D} = \mathbf{0}$. Position of 0s indicate parameter identifiable.

  $\boldsymbol{\alpha} = [-\psi/p \quad 1]$ (no parameters identifiable)
- Can find estimable parameter combination by solving PDEs

$$\sum_{i=1}^{q} \alpha_{ij} \frac{\partial f}{\partial \theta_i} = 0, j = 1, \ldots, q - r \qquad -\frac{\partial f}{\partial \psi}\frac{\psi}{p} + \frac{\partial f}{\partial p} = 0$$

solution shows can estimate ψp.

Symbolic Method (Occupancy Example)

- Also able to generalise results, e.g. in occupancy modelling robust design involves visiting a site more than once in a short space of time (within a season). Can estimate both parameters if there are at least two samples per season. Can extend results to multiple seasons.
- Requires symbolic algebra package, e.g. Maple, in most models to find rank.
- Can run out of memory finding rank in complex models.
- Extended Symbolic method (Cole, et al. 2010) uses reparameterisation to find simpler exhaustive summaries. For example: multi-state models with unobservable states (Cole, 2012).

Numerical Methods (Occupancy Example)

- Hessian Method (Viallefont et al, 1998):
- Hessian matrix should be singular and therefore have at least one Eigenvalue that is 0. Found numerically the Hessian is not necessarily singular.
- Hessian: $\boldsymbol{h} = \begin{bmatrix} 180.25 & 179.99 \\ 179.99 & 179.74 \end{bmatrix}$
- Standardised Eigenvalues: 1, -0.000003
- Eigenvalue close to zero, model is parameter redundant.
- Inaccurate - can give wrong result (see Cole and Morgan, 2010)
- Not able to find identifiable parameter or estimable parameter combinations.
- Not able to generalise results.
- But can be added to a software package (e.g. Mark).

Other Numerical Methods

- Simulation (Gimenez et al., 2004)
  - Simulate large data sets and compare bias and CV.
  - Can be inaccurate.
- Likelihood Profile (Gimenez et al., 2004)
  - Flat profile indicates parameter not identifiable.
  - Difficult to distinguish between parameter redundancy and nearly redundancy.
- Data Cloning (Lele et al., 2007, 2010)
  - Bayesian MCMC with uninformative priors and K clones of data set.
  - Variance $\Rightarrow 0$ as $K \Rightarrow \infty$ if parameter identifiable.
  - Can be inaccurate, especially when calculating whether a particular parameter is identifiable.
- Methods can be used to find individually identifiable parameters.
- None of these methods can find general results and estimable parameter combinations.

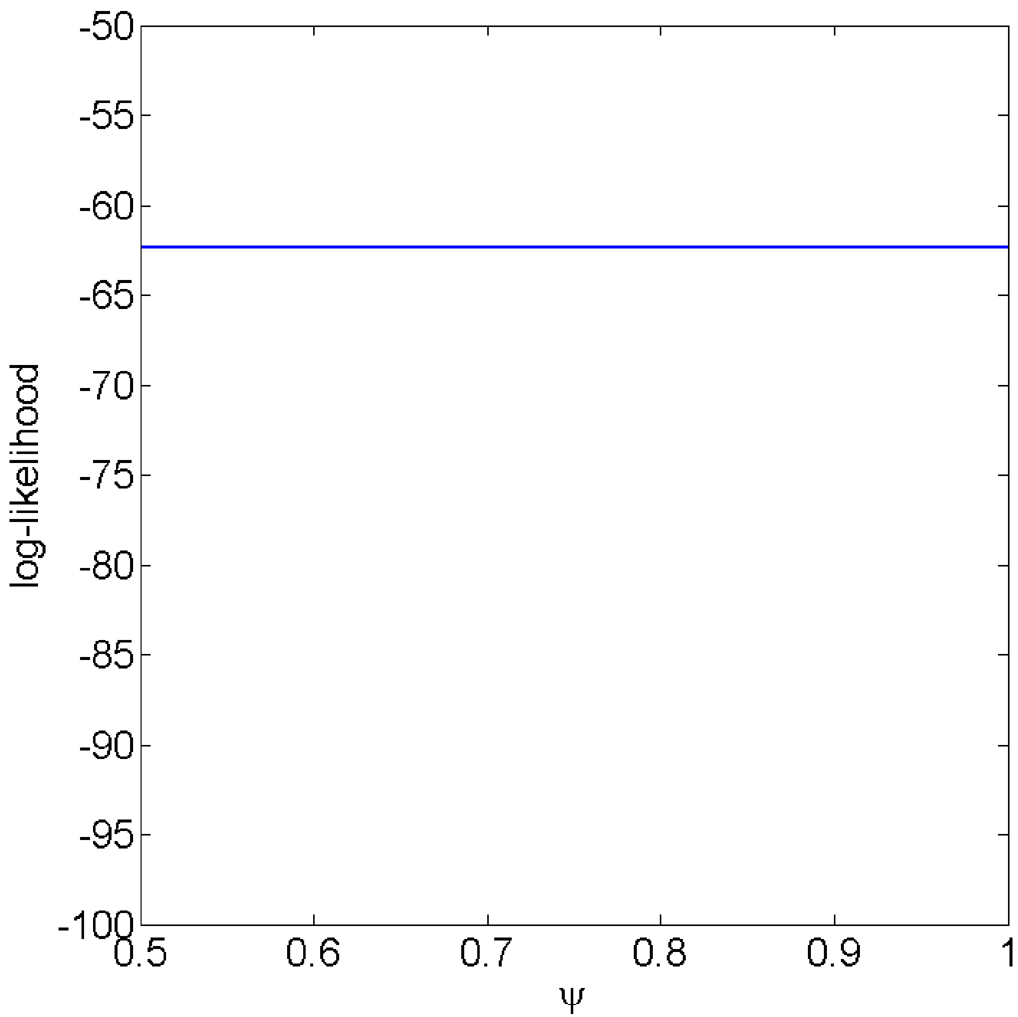
Hybrid Symbolic-Numeric Method

- Choquet and Cole (2012)
- Find derivative matrix, ***D***, symbolically (or using automatic differentiation).
- Find the rank of ***D*** at 5 random points, and the model rank is the maximum of all 5 ranks.
- Occupancy model: rank 1 in all 5 cases, therefore parameter redundant.
- Can also show if any of the original parameters are identifiable, but not the estimable parameter combinations.
- Can be added to a software package (e.g. E-Surge and M-Surge).
- Can this method be extended to find estimable parameter combinations and general results?

Comparison of Methods

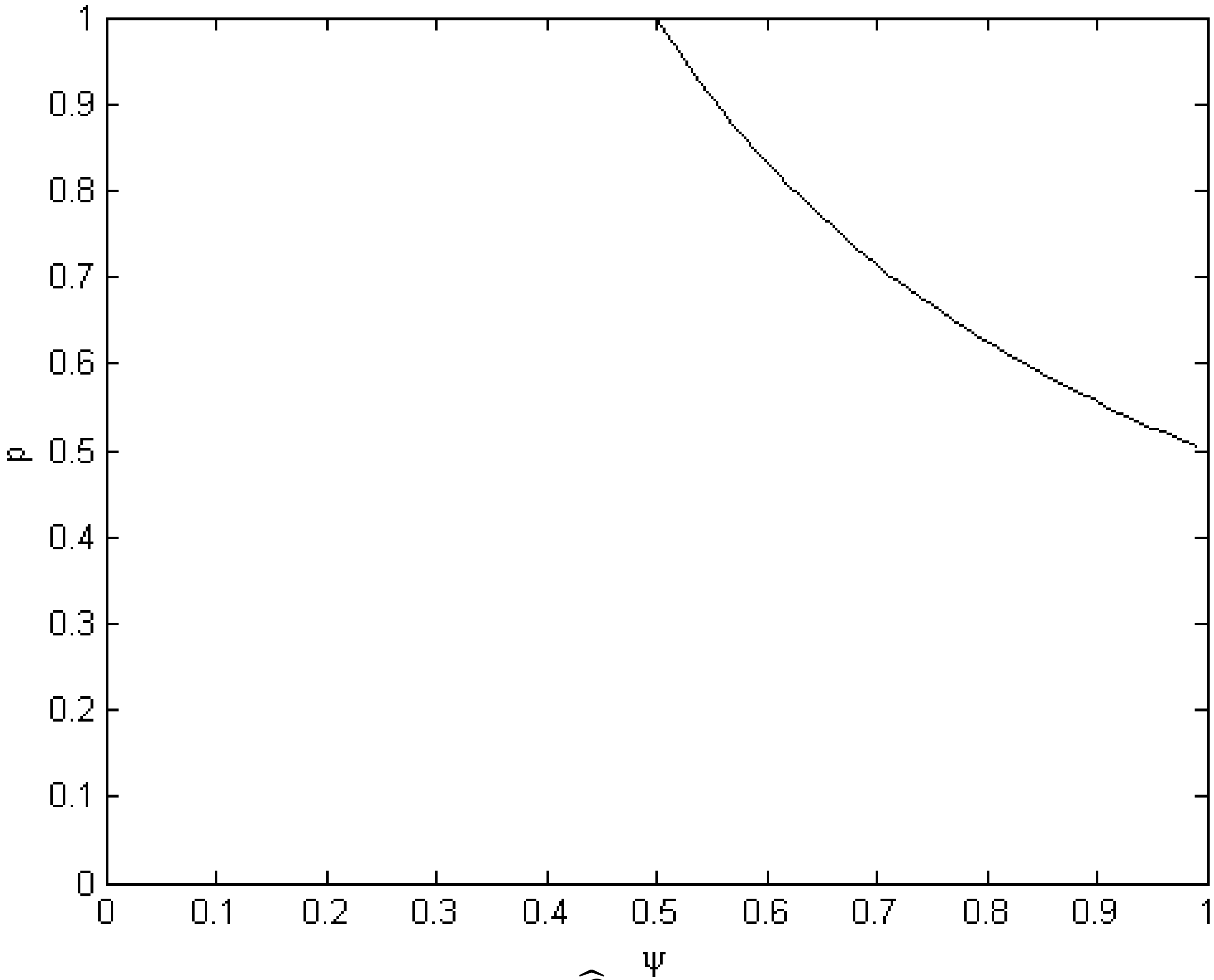
Method00	Accurate	Identifiable Parameters	Estimable Parameter Combinations	General Results	Complex Models	Automatic
Hessian	×	×	×	×	✓	✓
Simulation	×	✓	×	×	Slow	×
Lik. Profile	×	✓	×	×	✓	×
Data Cloning	×	✓	×	×	Slow	×
Symbolic	✓	✓	✓	✓	×	×
Ext. Symbolic	✓	✓	✓	✓	✓	×
Hybrid	✓	✓	×	×	✓	✓
Ext. Hybrid	**✓**	**✓**	**?**	**?**	**✓**	**?**

Occupancy Model - Likelihood Profile



Flat profile = Parameter Redundant

Occupancy Model - Likelihood Profile



Line is $p = \frac{\hat{\beta}}{\psi} = \frac{0.5}{\psi} \Rightarrow 0.5 = \psi p$

Profile gives the estimable parameter combinations

Extended Hybrid Method – Subset profiling

- Eisenberg and Hayashi (2014) use subset profiling to identify estimable parameter combinations manually in compartment modelling.
- Combine hybrid method (Choquet and Cole, 2012) with subset profiling (Eisenberg and Hayashi, 2014) and extend to automatically detect relationships between confounded parameters.

Step 1: Check whether model is parameter redundant using hybrid method and remove any parameters that are individually identifiable.

Step 2: Find subsets of parameters that are *nearly full rank*. (Have $d = q - r = 1$, and all subsets have $d = 0$).

Step 3: For each subset, fix all the parameters not in the subset. Then for each pair of parameters, in the subset, produce a likelihood profile. Fix one parameter, $\theta_{1,i}$, and store the MLE of the other parameter(s) in the subset, $\theta_{2,i}, \theta_{3,i}, \ldots$ $(i = 1, \ldots, n)$.

Extended Hybrid Method – Subset profiling

Step 4: Identify the relationships. Find $\beta_1, \beta_2, \beta_3, \beta_4$ that minimise

$$\sum_{i=1}^{n} \left(\theta_{2,i} - \frac{\beta_1 + \beta_2 \theta_{1,i}}{\beta_3 + \beta_4 \theta_{1,i}} \right)^2$$

to identify most common relationships (should equal 0).

Occupancy Example: $\theta_1 = \psi, \theta_2 = p, \beta_1 = 0.5, \beta_2 = 0, \beta_3 = 0, \beta_4 = 1$

$$p - \frac{0.5 + 0 \times \psi}{0 + 1 \times \psi} = p - \frac{0.5}{\psi} = 0 \Rightarrow p\psi = 0.5$$

Step 5: Turn relationships between pairs of parameters to form estimable parameter combinations.

Steps 1-4 can be programed to be automatic, step 5 not yet.

Mark-Recovery Example

- Animals marked and recovered dead. Mallards 1963-65:

$$\text{No. marked:} \begin{bmatrix} 962 \\ 702 \\ 1132 \end{bmatrix} \text{No. Recovered:} \begin{bmatrix} 82 & 35 & 18 \\ & 103 & 21 \\ & & 82 \end{bmatrix}$$

- Probability animals marked in year i and recovered in year j:

$$\boldsymbol{P} = \begin{bmatrix} (1-\phi_{1,1})\lambda_1 & \phi_{1,1}(1-\phi_2)\lambda_2 & \phi_{1,1}\phi_2(1-\phi_3)\lambda_3 \\ & (1-\phi_{1,2})\lambda_1 & \phi_{1,2}(1-\phi_2)\lambda_2 \\ & & (1-\phi_{1,3})\lambda_1 \end{bmatrix}$$

- $\phi_{1,t}$ probability animal survives 1st year at time t.
- ϕ_i probability an animal age i survives their ith year.
- λ_i probability animal aged i is recovered dead.
- Model is parameter redundant with estimable parameter combinations $\phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \lambda_1, (1-\phi_2)\lambda_2, \phi_2(1-\phi_3)\lambda_3$ (result found symbolically in Cole *et al*, 2012).
- $\phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \lambda_1$ – individually identifiable (can still be estimated).

Mark-Recovery Example

Model has 8 parameters $\boldsymbol{\theta} = [\phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \phi_2, \phi_3, \lambda_1, \lambda_2, \lambda_3]$

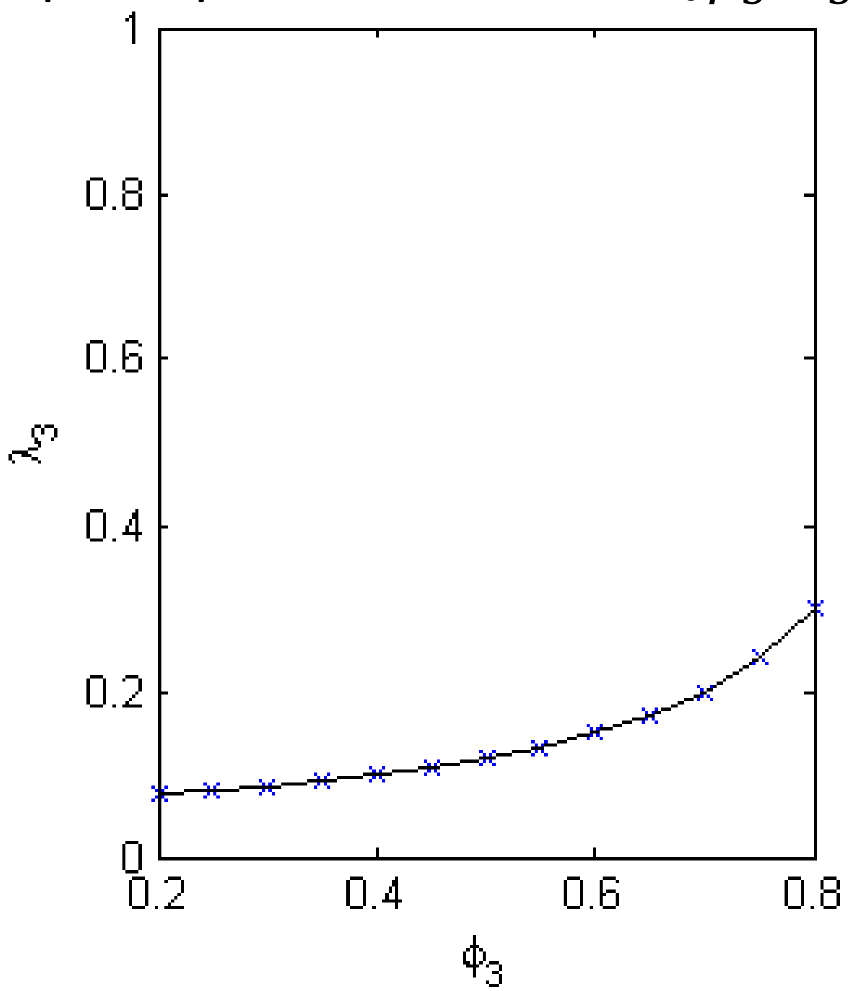
Step 1: Hybrid method can be use to show model is parameter redundant with rank 6. $\phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \lambda_1$ are individually identifiable. Non-identifiable parameters are $\phi_2, \phi_3, \lambda_2, \lambda_3$.

Step 2: Hybrid method used to find nearly full rank subsets:

$\{\phi_3, \lambda_3\}$,$\{\phi_2, \phi_3, \lambda_2\}$, $\{\phi_2, \lambda_2, \lambda_3\}$

Mark-Recovery Example

Step 3: x profiles for subset $\{\phi_3, \lambda_3\}$:

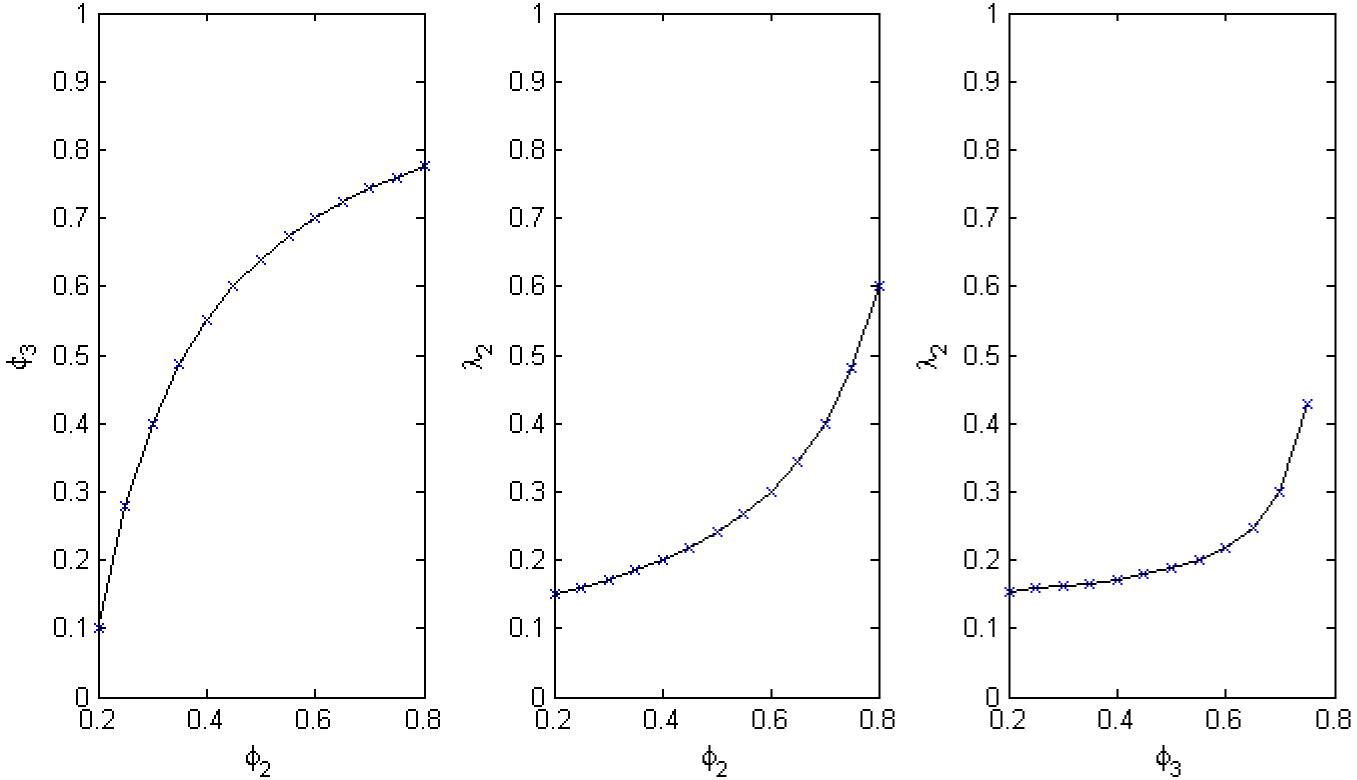


Step 4: ——— fitted relationship:

$$\lambda_3 - \frac{0.0736 - 0.0000\phi_3}{1.2267 - 1.2268\phi_3} = 0 \Rightarrow (1 - \phi_3)\lambda_3 = 0.06$$

Mark-Recovery Example

Step 3: x profiles for subset $\{\phi_2, \phi_3, \lambda_2\}$



Step 4: ——— fitted relationships:

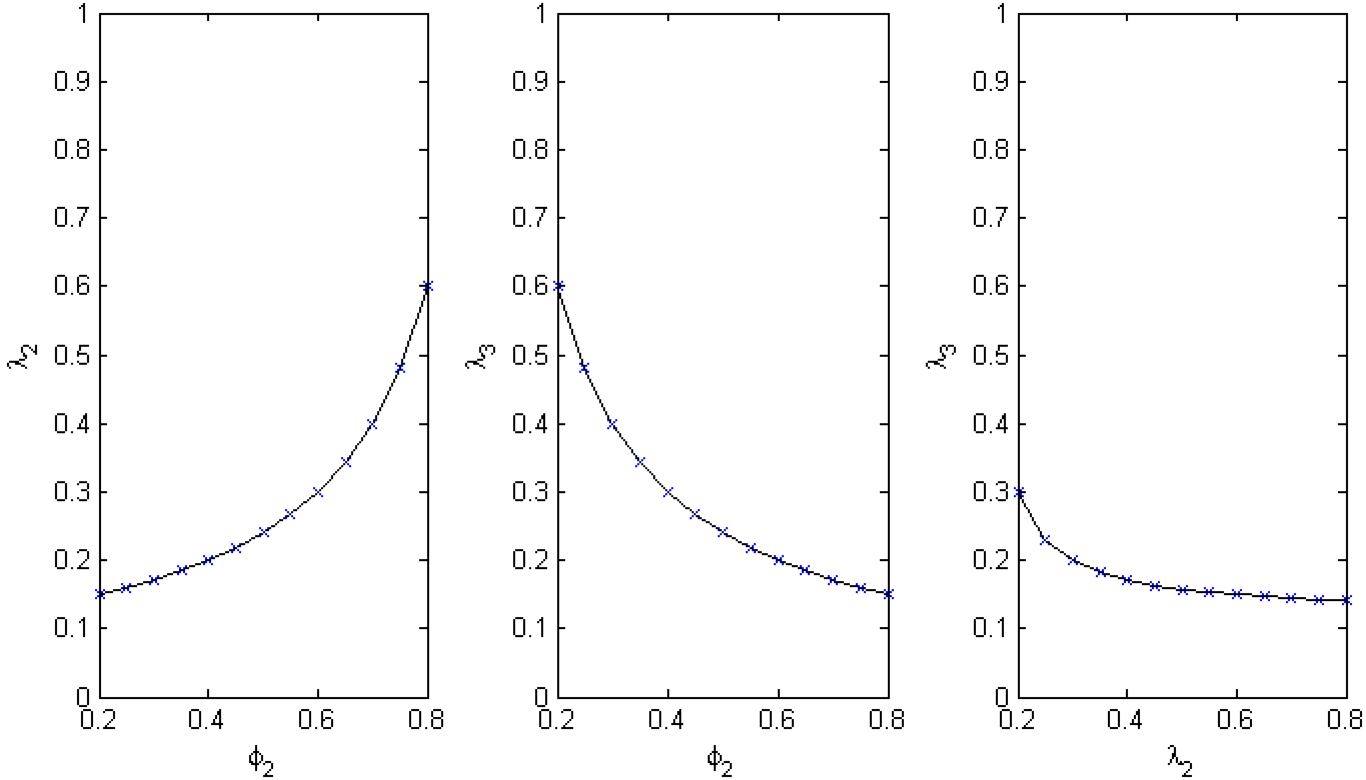
$$\phi_3 - \frac{-0.1800+0.9999\phi_2}{0.0000+0.9999\phi_2} = 0 \Rightarrow \phi_2(1-\phi_3) = 0.18$$

$$\lambda_2 - \frac{0.1200+0.0000\phi_2}{0.9999-0.9999\phi_2} = 0 \Rightarrow (1-\phi_2)\lambda_2 = 0.12$$

$$\lambda_2 - \frac{0.0013-0.0013\phi_3}{0.0088-0.0106\phi_3} = 0 \Rightarrow \frac{\lambda_2(0.83-\phi_3)}{1-\phi_3} = 0.12 \text{ (complex)}$$

Mark-Recovery Example

Step 3: x profiles for subset $\{\phi_2, \lambda_2, \lambda_3\}$



Step 4: ——— fitted relationships:

$$\lambda_2 - \frac{0.1200 + 0.0000\phi_2}{0.9999 - 0.9999\phi_2} = 0 \Rightarrow (1 - \phi_2)\lambda_2 = 0.12$$

$$\lambda_3 - \frac{(0.1472 - 0.0000\phi_2)}{0.0000 + 1.2267\phi_2} = 0 \Rightarrow \phi_2\lambda_3 = 0.12$$

$$\lambda_3 - \frac{-0.0000 + 0.0110\lambda_2}{-0.0110 + 0.0913\lambda_2} = 0 \Rightarrow \frac{\lambda_3}{\lambda_2}(0.12 - \lambda_2) = -0.12 \text{ (complex)}$$

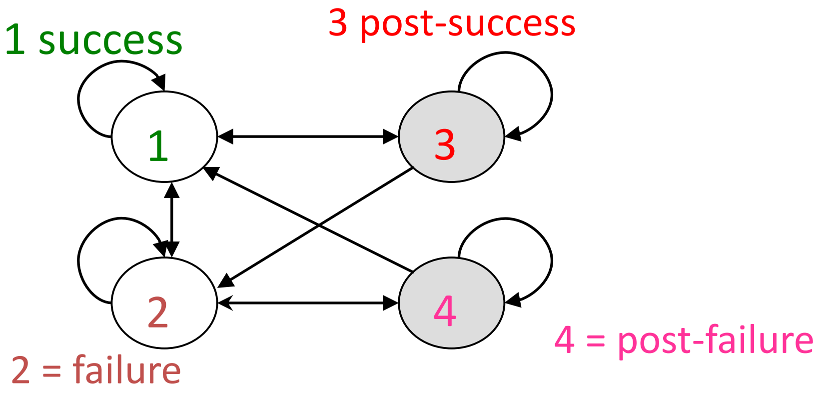
Mark-Recovery Example

Step 5:

- There could be more than one way of parameterising the estimable parameter combinations. Looking for the simplest.
- From step 1 also can deduce we need to find 2 combinations of parameters (rank 6, 4 identifiable parameters).
- $\phi_2(1-\phi_3) = 0.18,\ \phi_2\lambda_3 = 0.12,\ (1-\phi_3)\lambda_3 = 0.06$ are consistent with: $\phi_2(1-\phi_3)\lambda_3$
- $(1-\phi_2)\lambda_2 = 0.12$ is consistent with: $(1-\phi_2)\lambda_2$
- Estimable parameter combinations: $\phi_2(1-\phi_3)\lambda_3$ & $(1-\phi_2)\lambda_2$
- The other (complex) relationships can be shown (with a bit of algebra) to be consistent with estimable parameter combinations.

Multi-State Mark Recovery Example

- Hunter and Caswell (2009) examine parameter redundancy of multi-state mark-recapture models, but cannot evaluate the symbolic rank of the derivative matrix, so use a version of the hybrid method.
- Cole et al (2010) and Cole (2012) developed an extend symbolic method.
- 4 state breeding success model:



Wandering Albatross

survival — breeding given survival — successful breeding — recapture

$$\Phi = \begin{bmatrix} \sigma_1\beta_1\gamma_1 & \sigma_2\beta_2\gamma_2 & \sigma_3\beta_3\gamma_3 & \sigma_4\beta_4\gamma_4 \\ \sigma_1\beta_1(1-\gamma_1) & \sigma_2\beta_2(1-\gamma_2) & \sigma_3\beta_3(1-\gamma_3) & \sigma_4\beta_4(1-\gamma_4) \\ \sigma_1(1-\beta_1) & 0 & \sigma_3(1-\beta_3) & 0 \\ 0 & \sigma_2(1-\beta_2) & 0 & \sigma_4(1-\beta_4) \end{bmatrix}$$

$$\Pi = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

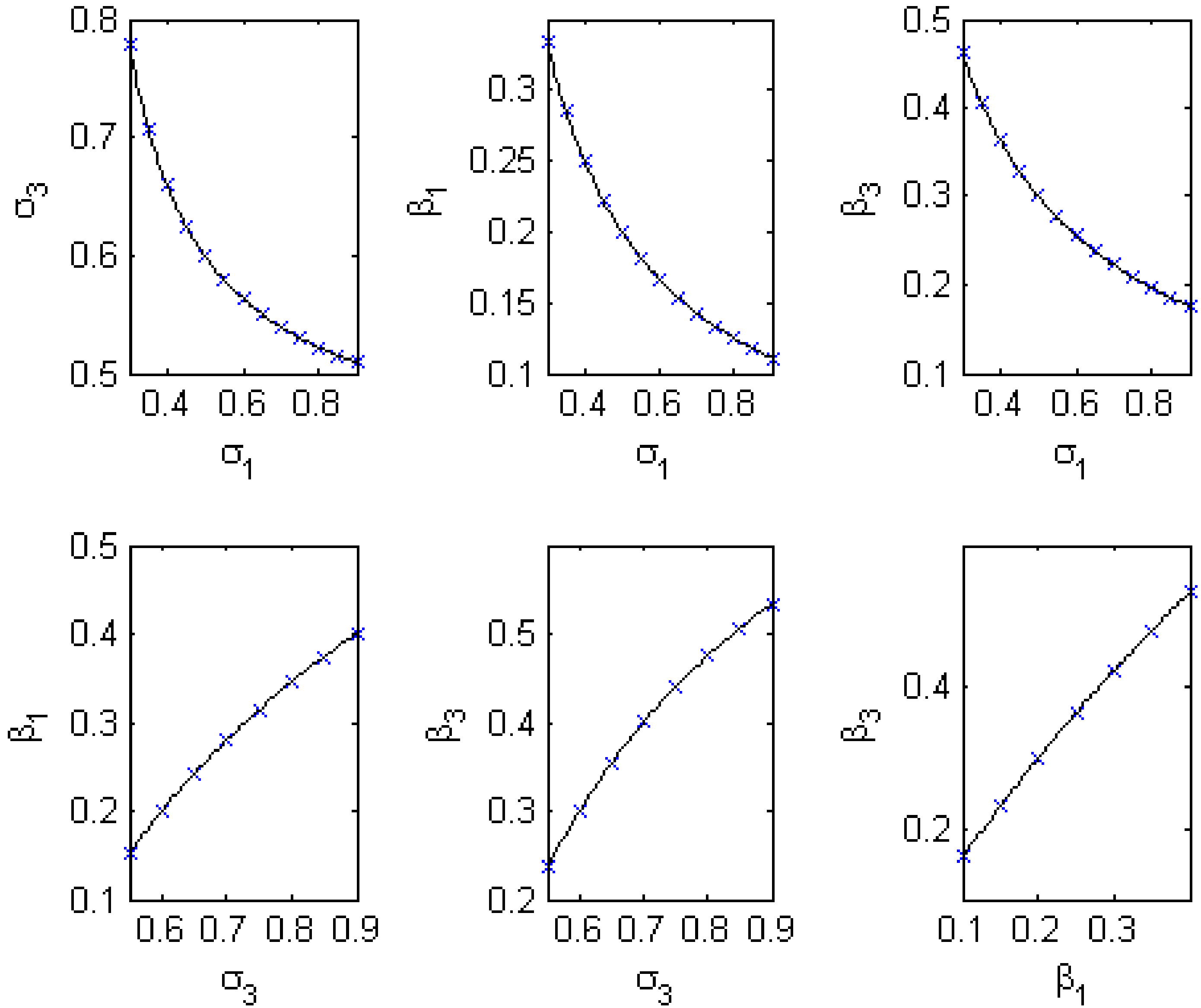
Multi-State Mark Recovery Example

- Model has 14 parameters
- $\boldsymbol{\theta} = [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \beta_1, \beta_2, \beta_3, \beta_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4, p_1, p_2]$
- The hybrid method can be used to show the rank is 12. Therefore the model is parameter redundant.
- The hybrid method can also be used to show that the parameters $\gamma_1, \gamma_2, \gamma_3, \gamma_4, p_1, p_2$ are identifiable.
- Nearly full rank subsets are:

$$\{\sigma_1, \sigma_3, \beta_1, \beta_3\}, \{\sigma_2, \sigma_4, \beta_2, \beta_4\}$$

Multi-State Mark Recovery Example

Subset $\{\sigma_1, \sigma_3, \beta_1, \beta_3\}$



$$\sigma_3 = \frac{(-0.3 - 4.2\sigma_1)}{1 - 10\sigma_1}$$

$$\beta_1 = \frac{0.1}{\sigma_1}$$

$$\beta_3 = \frac{2.4}{1 + 14\sigma_1}$$

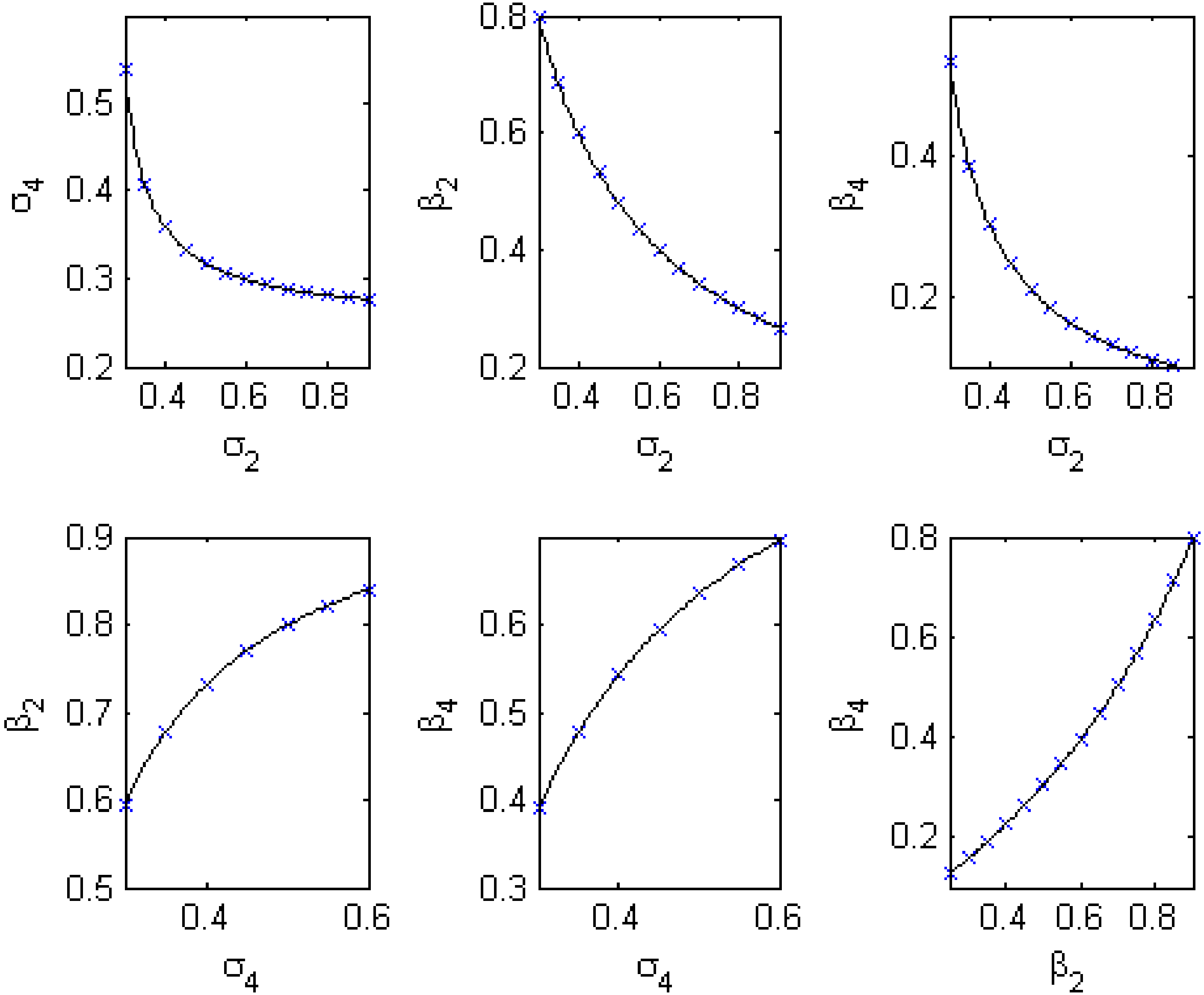
$$\beta_1 = \frac{-0.42 + \sigma_3}{0.3 + \sigma_3}$$

$$\beta_3 = \frac{-0.42 + \sigma_3}{\sigma_3}$$

$$\beta_3 = \frac{1.7160\beta_1}{1 + 0.7160\beta_1}$$

Multi-State Mark Recovery Example

Subset $\{\sigma_2, \sigma_4, \beta_2, \beta_4\}$



$$\sigma_4 = \frac{(0.18 - 1.05\sigma_2)}{1 - 4.1725\sigma_2}$$

$$\beta_2 = \frac{0.24}{\sigma_2}$$

$$\beta_4 = \frac{0.2963}{-0.73 + 4.27\sigma_2}$$

$$\beta_2 = \frac{-0.18 + \sigma_4}{-0.10 + \sigma_4}$$

$$\beta_4 = \frac{-0.18 + \sigma_4}{\sigma_4}$$

$$\beta_4 = \frac{0.4367\beta_2}{1 - 0.5633\beta_2}$$

Multi-State Mark Recovery Example

- $\beta_1 = \frac{0.1}{\sigma_1} \Rightarrow \sigma_1\beta_1 = 0.1$
- $\beta_3 = \frac{-0.42+\sigma_3}{\sigma_3} \Rightarrow \sigma_3(1-\beta_3) = 0.42$
- $\beta_3 = \frac{1.7160\beta_1}{1+0.7160\beta_1} \Rightarrow \beta_3 = \frac{1.7160\beta_1}{1+1.7160\beta_1-\beta_1} \Rightarrow \frac{\beta_3(1-\beta_1)}{\beta_1(1-\beta_3)} = 1.7160$
- $\beta_2 = \frac{0.24}{\sigma_2} \Rightarrow \sigma_2\beta_2 = 0.24$
- $\beta_4 = \frac{-0.18+\sigma_4}{\sigma_4} \Rightarrow \sigma_4(1-\beta_4) = 0.18$
- $\beta_4 = \frac{0.4367\beta_2}{1-0.5633\beta_2} \Rightarrow \beta_4 = \frac{0.4367\beta_2}{1+0.4367\beta_2-\beta_2} \Rightarrow \frac{\beta_4(1-\beta_2)}{\beta_2(1-\beta_4)} = 0.4367$
- Estimable parameter combinations:

  $\sigma_1\beta_1, \sigma_3(1-\beta_3), \frac{\beta_3(1-\beta_1)}{\beta_1(1-\beta_3)}, \sigma_2\beta_2, \sigma_4(1-\beta_4), \frac{\beta_4(1-\beta_2)}{\beta_2(1-\beta_4)}$
- (Results are a reparameterisation of those found using extended symbolic method.)

General Results

- In symbolic method general results found using the extension theorem (Catchpole and Morgan, 1997, Cole et al, 2010).
- This can be extended to the Hybrid method.

Mark Recovery Example:

Reparameterise in terms of estimable parameters, $\phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \lambda_1, b_1 = (1-\phi_2)\lambda_2$ and $b_2 = \phi_2(1-\phi_3)\lambda_3$.

Hybrid Method shows we can estimate $\phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \lambda_1, \beta_1, \beta_2$.

Add an extra year of ringing and recovery adds extra parameters $\phi_{1,4}$ and $b_3 = \phi_2\phi_3(1-\phi_4)\lambda_4$.

Hybrid method shows can also estimate $\phi_{1,4}, b_3$.

By the extension theorem, for *n* years of ringing and recovery, model is always parameter redundant, estimable parameter combinations will be

$$\phi_{1,1}, \phi_{1,2}, \dots, \phi_{1,n}, \lambda_1, (1-\phi_2)\lambda_2, \phi_2(1-\phi_3)\lambda_3, \dots, \left(\prod_{i=2}^{n-1} \phi_i\right)(1-\phi_n)\lambda_n$$

Conclusion

Method	Accurate	Identifiable Parameters	Estimable Parameter Combinations	General Results	Complex Models	Automatic
Hessian	×	×	×	×	✓	✓
Simulation	×	✓	×	×	Slow	×
Lik. Profile	×	✓	×	×	✓	×
Data Cloning	×	✓	×	×	Slow	×
Symbolic	✓	✓	✓	✓	×	×
Ext. Symbolic	✓	✓	✓	✓	✓	×
Hybrid	✓	✓	×	×	✓	✓
Ext. Hybrid	✓	✓	✓	✓	✓	**Semi**

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