

# Properties of orthogonal polynomials

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## Definition

A polynomial  $R_n$ ,  $\deg R_n = n$ ,  $n \geq r$  is quasi-orthogonal of order  $r$  where  $n, r \in \mathbb{N}$  with respect to  $w(x) > 0$  on  $I$  if

$$\int_I x^k R_n(x) w(x) dx \begin{cases} = 0 & \text{for } k = 0, 1, \dots, n-r-1 \\ \neq 0 & \text{for } k = n-r. \end{cases}$$

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A characterisation of quasi-orthogonality of order  $r$ :

## Theorem (Shohat)

Let  $\{P_n\}_{n=0}^{\infty}$  be a family of orthogonal polynomials with respect to  $w(x) > 0$  on  $[a, b]$ . Then the  $n$ -th degree polynomial  $R_n$  is quasi-orthogonal of order  $r$  on  $[a, b]$  with respect to  $w(x)$  if and only if there exist constants  $c_i$ ,  $i = 0, \dots, r$  and  $c_0 c_r \neq 0$  such that

$$R_n(x) = c_0 P_n(x) + c_1 P_{n-1}(x) + \dots + c_r P_{n-r}(x).$$

- Riesz [1923]: Quasi-orthogonal polynomials of order 1

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MARCEL RIESZ

- Riesz [1923]: Quasi-orthogonal polynomials of order 1



Figure: Marcel and Frigyes Riesz



Figure: Lipót Fejér

- Fejér [1933]: Quasi-orthogonality of order 2
- Shohat [1937]: Quasi-orthogonality of any order  $r$



- Chihara [1957]: Generalised definition of quasi-orthogonal polynomials and studied them in the context of three-term recurrence relations
- Dickinson [1961]: System of recurrence relations necessary and sufficient for quasi-orthogonality of order 1
- Draux [1990]: Proved the converse of one of Chihara's results
- Brezinski, Driver, Redivo-Zaglia [2004]: Results on the real zeros of quasi-orthogonal polynomials
- Joulak [2005]: Extended these results by giving necessary and sufficient conditions

# A result by Dickinson

Dickinson applied systems of recurrence relations that are necessary and sufficient for quasi-orthogonality to some special cases of Fasenmyer polynomials

$$f_n(a, x) = {}_3F_2 \left( \begin{matrix} -n, n+1, a \\ \frac{1}{2}, 1 \end{matrix} ; x \right) = \sum_{m=0}^n \frac{(-n)_m (n+1)_m (a)_m}{\left(\frac{1}{2}\right)_m (1)_m} \frac{x^m}{m!}.$$

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## Theorem (Dickenson, 1961)

*The polynomials  $f_n\left(\frac{3}{2}, x\right)$  and  $f_n(2, x)$  are quasi-orthogonal of order 1 on the interval  $(0, 1)$  with weights  $(1-x)$  and  $x^{-1/2}(1-x)^{3/2}$  respectively.*

These turn out to be very special cases of more general classes of quasi-orthogonal  ${}_pF_q$  polynomials arising from orthogonal  ${}_{p-1}F_{q-1}$  polynomials (cf. Johnston and J [2015]).



Figure: Celine Fasenmyer

# Zeros of quasi-orthogonal polynomials of order $r$

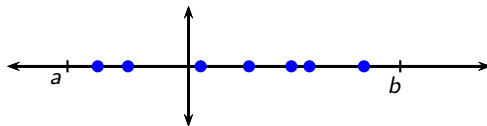
## Theorem (Shohat)

*If  $R_n$  is quasi-orthogonal of order  $r$  on  $[a, b]$  with respect to a positive weight function, then at least  $(n - r)$  distinct zeros of  $R_n$  lie in the interval  $(a, b)$*

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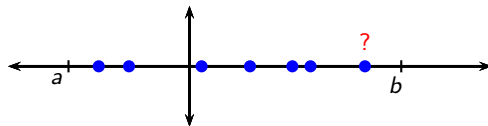
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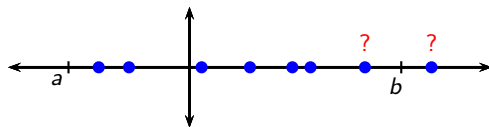


Figure: Quasi-orthogonality of order 1: at least  $n - 1$  zeros in interval  $I$



# Quasi-orthogonal polynomials of order 2

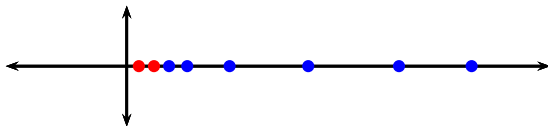


Figure: Quasi-orthogonality of order 2: at least  $n - 2$  zeros in interval  $I$

# Quasi-orthogonal polynomials of order 2

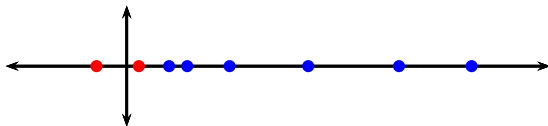


Figure: Quasi-orthogonality of order 2: at least  $n - 2$  zeros in interval  $I$

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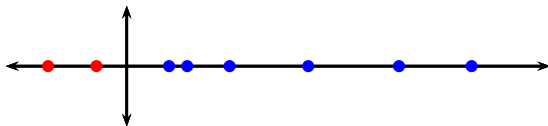


Figure: Quasi-orthogonality of order 2: at least  $n - 2$  zeros in interval  $I$

# Quasi-orthogonal polynomials of order 2

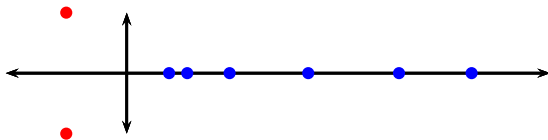


Figure: Quasi-orthogonality of order 2: at least  $n - 2$  zeros in interval  $I$

# Semiclassical orthogonal polynomials

Al-Salam and Chihara [1972] showed that orthogonal polynomial sets satisfying

$$\pi(x)P_n'(x) = (a_n x + b_n)P_n(x) + c_n P_{n-1}$$

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Askey raised the more general question of what orthogonal polynomial sets have the property that their derivatives satisfy a relation of the form

$$\pi(x)P_n'(x) = \sum_{k=n-t}^{n+s} \alpha_{nk} P_k(x).$$

This problem was considered by Shohat [1939] and later, independently, by Freud [1976], as well as Bonan and Nevai [1984].

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Maroni [1985] stated the problem in a different way, trying to find all orthogonal polynomial sets whose derivatives are quasi-orthogonal, and called such orthogonal polynomial sets semi-classical.

# Semiclassical orthogonal polynomials

Consider the Pearson equation satisfied by the weight  $w(x)$

$$\frac{d}{dx}[\sigma(x)w(x)] = \tau(x)w(x)$$

Classical orthogonal polynomials:  $\sigma(x)$  and  $\tau(x)$  are polynomials with  $\deg(\sigma) \leq 2$  and  $\deg(\tau) = 1$

	$w(x)$	$\sigma(x)$	$\tau(x)$
Hermite	$\exp(-x^2)$	1	$-2x$
Laguerre	$x^\alpha \exp(-x)$	$x$	$1 + \alpha - x$
Jacobi	$(1-x)^\alpha (1+x)^\beta$	$(1-x)^2$	$\beta - \alpha - (2 + \alpha + \beta)x$

Semi-classical orthogonal polynomials:  $\sigma(x)$  and  $\tau(x)$  are polynomials with either  $\deg(\sigma) > 2$  or  $\deg(\tau) > 1$

	$w(x)$	$\sigma(x)$	$\tau(x)$
semi-classical Laguerre	$x^\lambda \exp(-x^2 + tx)$	$x$	$1 + \lambda + tx - 2x^2$
generalized Freud	$ x ^{2\lambda+1} \exp(-x^4 + 2tx^2)$	$x$	$2\lambda + 2 - 2tx^2 - x^4$



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## 18.32 OP's with Respect to Freud Weights

A *Freud weight* is a weight function of the form

$$w(x) = \exp(-Q(x)), \quad -\infty < x < \infty$$

where  $Q(x)$  is real, even, nonnegative, and continuously differentiable. Of special interest are the cases  $Q(x) = x^{2m}$ ,  $m = 1, 2, \dots$ . **No explicit expressions for the corresponding OP's are available.** However, for asymptotic approximations in terms of elementary functions for the OP's, and also for their largest zeros, see Levin and Lubinsky [2001] and Nevai [1986]. For a uniform asymptotic expansion in terms of Airy functions for the OP's in the case  $x^4$  see Bo and Wong [1999].

# The generalized Freud weight

Expressions for the recurrence coefficients associated with the semi-classical Laguerre weight

$$w(x) = x^\lambda \exp(-x^2 + tx), \quad x \in (0, \infty) \text{ for } \lambda > -1 \text{ and } t \in \mathbb{R}$$

and the generalized Freud weight

$$w(x) = |x|^{2\lambda+1} \exp(-x^4 + tx^2) \quad x \in \mathbb{R} \text{ for } \lambda > -1 \text{ and } t \in \mathbb{R}$$

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The link between the theory of Painlevé equations and orthogonal polynomials is given by the moments of the weight

$$\mu_n = \int_a^b x^n w(x) dx, \quad n = 0, 1, 2, \dots$$

which allow the Hankel determinant to be written as a Wronskian.

# Moments of the generalized Freud weight

The first moment,  $\mu_0(t; \lambda)$ , can be obtained using the integral representation of a parabolic cylinder function.

$$\begin{aligned}\mu_0(t; \lambda) &= \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(-x^4 + tx^2) dx \\ &= \frac{\Gamma(\lambda+1)}{2^{(\lambda+1)/2}} \exp\left(\frac{1}{8}t^2\right) D_{-\lambda-1}\left(-\frac{1}{2}\sqrt{2}t\right).\end{aligned}$$

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The even moments are

$$\begin{aligned}\mu_{2n}(t; \lambda) &= \int_{-\infty}^{\infty} x^{2n} |x|^{2\lambda+1} \exp(-x^4 + tx^2) dx \\ &= \mu_0(t; \lambda + n) = \frac{d^n}{dt^n} \mu_0(t, \lambda), \quad n = 1, 2, \dots\end{aligned}$$

whilst the odd ones are

$$\mu_{2n+1}(t; \lambda) = \int_{-\infty}^{\infty} x^{2n+1} |x|^{2\lambda+1} \exp(-x^4 + tx^2) dx = 0, \quad n = 1, 2, \dots$$

since the integrand is odd.

# The recurrence coefficients

Monic orthogonal polynomials with respect to the generalized Freud weight

$$|x|^{2\lambda+1} \exp(-x^4 + tx^2)$$

satisfy the three-term recurrence relation

$$xS_n(x; t) = S_{n+1}(x; t) + \beta_n(t; \lambda)S_{n-1}(x; t)$$

where  $\beta_n(t; \lambda) > 0$ ,  $S_{-1}(x; t) = 0$ ,  $S_0(x; t) = 1$ ,  $\beta_0(t; \lambda) = 0$   
and

$$\begin{aligned}\beta_1(t; \lambda) &= \frac{\mu_2(t; \lambda)}{\mu_0(t; \lambda)} = \frac{\int_{-\infty}^{\infty} x^2 |x|^{2\lambda+1} \exp(-x^4 + tx^2) dx}{\int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(-x^4 + tx^2) dx} \\ &= \frac{1}{2}t + \frac{1}{2}\sqrt{2} \frac{D_{-\lambda}\left(-\frac{1}{2}\sqrt{2}t\right)}{D_{-\lambda-1}\left(-\frac{1}{2}\sqrt{2}t\right)},\end{aligned}$$

...



# The differential-difference equation

The coefficients  $A_n(x; t)$  and  $B_n(x; t)$  in the relation

$$\frac{dP_n}{dx}(x; t) = -B_n(x; t)P_n(x; t) + A_n(x; t)P_{n-1}(x; t), \quad (1)$$

satisfied by semi-classical orthogonal polynomials are of interest since differentiation of this differential-difference equation yields the second order differential equation satisfied by the orthogonal polynomials.

- Shohat [1939] gave a procedure using quasi-orthogonality to derive (1) for weights  $w(x; t)$  such that  $w'(x; t)/w(x; t)$  is a rational function;
- This technique was rediscovered by several authors including Bonan, Freud, Mhaskar and Nevai approximately 40 years later;
- Method of ladder operators was introduced by Chen and Ismail [1997];
- Chen and Feigin [2006] adapt the method of ladder operators to the situation where the weight function vanishes at one point.
- Clarkson, J and Kelil [2016] generalize the work by Chen and Feigin, giving a more explicit expression for the coefficients in (1) when the weight function is positive on the real line except for one point.

The derivatives of monic orthogonal polynomials  $S_n(x; t)$  with respect to the generalized Freud weight are quasi-orthogonal of order  $m = 5$ .

$$x \frac{dS_n}{dx}(x; t) = \sum_{k=n-4}^n c_{n,k} S_k(x; t),$$

where the coefficient  $c_{n,k}$ , for  $n-4 \leq k \leq n$  and  $h_k > 0$ , is given by

$$c_{n,k} = \frac{1}{h_k} \int_{-\infty}^{\infty} \frac{dS_n(x; t)}{dx} x S_k(x; t) w(x; t) dx.$$

Integrating by parts, we obtain for  $n-4 \leq k \leq n-1$ ,

$$\begin{aligned} h_k c_{n,k} &= \left[ x S_k(x; t) S_n(x; t) w(x; t) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dx} [x S_k(x; t) w(x; t)] S_n(x; t) dx \\ &= - \int_{-\infty}^{\infty} \left[ S_n(x; t) S_k(x; t) + x S_n(x; t) \frac{dS_k(x; t)}{dx} \right] w(x; t) dx \\ &\quad - \int_{-\infty}^{\infty} x S_n(x; t) S_k(x; t) \frac{dw(x; t)}{dx} dx \\ &= - \int_{-\infty}^{\infty} x S_n(x; t) S_k(x; t) \frac{dw(x; t)}{dx} dx \\ &= - \int_{-\infty}^{\infty} S_n(x; t) S_k(x; t) \left( -4x^4 + 2tx^2 + 2\lambda + 1 \right) w(x; t) dx \\ &= \int_{-\infty}^{\infty} (4x^4 - 2tx^2) S_n(x; t) S_k(x; t) w(x; t) dx. \end{aligned}$$

Iterating the three-term recurrence relation, the following relations are obtained:

$$x^2 S_n = S_{n+2} + (\beta_n + \beta_{n+1})S_n + \beta_n \beta_{n-1} S_{n-2},$$

$$\begin{aligned} x^4 S_n &= S_{n+4} + (\beta_n + \beta_{n+1} + \beta_{n+2} + \beta_{n+3})S_{n+2} \\ &\quad + [\beta_n(\beta_{n-1} + \beta_n + \beta_{n+1}) + \beta_{n+1}(\beta_n + \beta_{n+1} + \beta_{n+2})] S_n \\ &\quad + \beta_n \beta_{n-1}(\beta_{n-2} + \beta_{n-1} + \beta_n + \beta_{n+1})S_{n-2} \\ &\quad + (\beta_n \beta_{n-1} \beta_{n-2} \beta_{n-3})S_{n-4}. \end{aligned}$$

Substituting these into

$$h_k c_{n,k} = \int_{-\infty}^{\infty} (4x^4 - 2tx^2) S_n(x; t) S_k(x; t) w(x; t) dx$$

yields the coefficients  $\{c_{n,k}\}_{k=n-4}^{n-1}$  in

$$x \frac{dS_n}{dx}(x; t) = \sum_{k=n-4}^n c_{n,k} S_k(x; t),$$

as follows:

$$c_{n,n-4} = 4\beta_n \beta_{n-1} \beta_{n-2} \beta_{n-3},$$

$$c_{n,n-3} = 0,$$

$$c_{n,n-2} = 4\beta_n \beta_{n-1} (\beta_{n-2} + \beta_{n-1} + \beta_n + \beta_{n+1} - \frac{1}{2}t),$$

$$c_{n,n-1} = 0.$$

Lastly, we consider the case when  $k = n$ .

A similar (but longer) argument and the fact that the recurrence coefficient  $\beta_n$  satisfies discrete P<sub>I</sub>

$$\beta_{n+1} + \beta_n + \beta_{n-1} = \frac{1}{2}t + \frac{2n + (2\lambda + 1)[1 - (-1)^n]}{8\beta_n}$$

yields

$$c_{n,n} = n.$$

Now we can write

$$x \frac{dS_n}{dx}(x; t) = c_{n,n-4}S_{n-4}(x; t) + c_{n,n-2}S_{n-2}(x; t) + c_{n,n}S_n(x; t). \quad (2)$$

In order to express  $S_{n-4}$  and  $S_{n-2}$  in terms of  $S_n$  and  $S_{n-1}$ , we iterate the recurrence relation.

# The differential-difference equation

Theorem (Clarkson, J & Kelil, 2016)

For the generalized Freud weight

$$w(x; t) = |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right), \quad x \in \mathbb{R}, \lambda > 0$$

the monic orthogonal polynomials  $S_n(x; t)$  satisfy

$$\frac{dS_n}{dx}(x; t) = -B_n(x; t)S_n(x; t) + A_n(x; t)S_{n-1}(x; t)$$

with

$$A_n(x; t) = 4\beta_n\left(x^2 - \frac{1}{2}t + \beta_n + \beta_{n+1}\right),$$

$$B_n(x; t) = 4x\beta_n + \frac{(2\lambda + 1)[1 - (-1)^n]}{2x}.$$

# The differential equation

Theorem (Clarkson, J & Kelil, 2016)

For the generalized Freud weight, the monic orthogonal polynomials  $S_n(x; t)$  satisfy the differential equation

$$\frac{d^2 S_n}{dx^2}(x; t) + R_n(x; t) \frac{d S_n}{dx}(x; t) + T_n(x; t) S_n(x; t) = 0,$$

$$R_n(x; t) = -4x^3 + 2tx + \frac{2\lambda + 1}{x} - \frac{2x}{x^2 - \frac{1}{2}t + \beta_n + \beta_{n+1}},$$

$$\begin{aligned} T_n(x; t) = & 4nx^2 + 4\beta_n + 16\beta_n(\beta_n + \beta_{n+1} - \frac{1}{2}t)(\beta_n + \beta_{n-1} - \frac{1}{2}t) \\ & + 4(2\lambda + 1)(-1)^n \beta_n - \frac{8\beta_n x^2 + (2\lambda + 1)[1 - (-1)^n]}{x^2 - \frac{1}{2}t + \beta_n + \beta_{n+1}} \\ & + (2\lambda + 1)[1 - (-1)^n] \left( t - \frac{1}{2x^2} \right). \end{aligned}$$

Note that the coefficients in the differential equation of the semiclassical orthogonal polynomials are not the same as those of the Pearson equation.



