Properties of orthogonal polynomials
Assignment 1

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Question 1

A set of polynomials $\{\varphi(x)\}$; $n = 0, 1; 2, \ldots,$ is called a *simple set* if $\varphi_n(x)$ is of degree precisely n in x so that t e set contains one polynomial of each degree. One immediate result of the definition of a simple set of polynomials is that any polynomials can be expressed linearly in terms of the elements of that simple set.

Theorem

If $\{\varphi_n(x)\}$ is a simple set of polynomials and if $P(x)$ is a polynomial of degree m, there exist constants c_k such that

$$P(x) = \sum_{k=0}^{m} c_k \varphi_k(x).$$

The c_k are functions of k and of any parameters involved in $P(x)$.

Prove this theorem.

Question 2

Prove the following theorem which gives an equivalent condition for orthogonality.

Theorem

If $\{p_n(x)\}$ frm a simple set of real polynomials and $w(x) > 0$ on $a < x < b$, a necessary and sufficient condition that the set $\{p_n(x)\}$ be orthogonal with respect to $w(x)$ over the interval $a < x < b$ is that

$$\int_a^b w(x)x^k p_n(x)dx = 0, \quad k = 0, 1, 2, ..., n-1 \tag{1}$$

The Christoffel-Darboux formula states that if $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials with respect to the weight function $w(x)$ on $[a, b]$ and the leading coefficient of $p_n(x)$ is k_n, then

$$\sum_{m=0}^{n} \frac{p_m(x)p_m(y)}{h_m} = \frac{k_n}{k_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{(x-y)h_n}.$$

Use the three term recurrence relation to prove the Christoffel Darboux formula and then derive the confluent form

$$\sum_{m=0}^{n} \frac{\{p_m(x)\}^2}{h_m} = \frac{k_n}{h_n\, k_{n+1}}(p'_{n+1}(x)p_n(x) - p_{n+1}(x)p'_n(x)),\ n = 0, 1, 2, \ldots$$

Question 4

Suppose $f(x)$ and $g(x)$ are n-times differentiable functions. Prove Leibnitz' formula for the n^{th} derivative of a product of two functions, given by

$$\frac{d^n}{dx^n}\{f(x)g(x)\} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)\, g^{(n-k)}(x)$$

where $f^{(k)}$ denotes $\dfrac{d^k f}{dx^k}$, i.e the k^{th} derivative of $f(x)$ with respect to x