TUTORIALS ON DISCRETE PAINLEVÉ EQUATIONS

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Tutorial 1

1. Consider the operators s_i, $i = 0, 1, 2$, π given in the lectures, which are defined by [3]

$$
\begin{aligned}
s_i(\alpha_i) &= -\alpha_i, & s_i(\alpha_j) &= \alpha_j + \alpha_i,\ j = i \pm 1 \\
s_i(f_i) &= f_i, & s_i(f_j) &= f_j \pm \frac{\alpha_i}{f_i},\ j = i \pm 1 \\
\pi(\alpha_i) &= \alpha_{i+1}, & \pi(f_i) &= f_{i+1},
\end{aligned}
$$

   for $i \in \mathbb{N} \bmod 3$.

   (a) Show that ${s_i}^2 = 1$, $i = 0, 1, 2$.

   (b) Define $g_0 = f_0$, $f_1 = g_1 - \alpha_0/g_0$, $f_2 = g_2 + \alpha_0/g_0$. If f_i satisfy the symmetric form of $\mathrm{P_{IV}}$, deduce the differential equations satisfied by g_i.

2. Consider the following form of dP1

$$c_3\, w_n(w_{n+1} + w_n + w_{n-1}) + c_2 w_n = c_1 + c_0(-1)^n - n.$$

   (a) Find the compatibility conditions for the linear system [1]

$$
\begin{aligned}
w_n\, \phi_{n+1} &= \lambda\, \phi_n - \phi_{n-1} \\
\frac{\partial \phi_n}{\partial \lambda} &= a_n\, \phi_{n+1} + b_n\, \phi_n
\end{aligned}
$$

   where a_n, b_n are functions of λ and n. (This system is called a Lax pair.)

   (b) Show that b_n can be eliminated from the compatibility conditions, and obtain a third-order difference equation for $p_n = a_n/w_n$ alone.

   (c) Assuming $p_n = \sum_{k=0}^{m} P_{2k,n}\, \lambda^{2k}$, find the coefficients $P_{2k,n}$ and show that the case $m = 1$ leads to dP1.

3. Consider the discrete equation

$$w_n(w_{n+1} + w_n + w_{n+1}) = z_n + c w_n$$

   where z_n is a given function of n and c is constant [2].

   (a) Consider the autonomous case $z_n = a$, where a is constant. If w_n is arbitrarily small, the equation gives an arbitrary large value for w_{n+1}. To study this more carefully, let $w_{-1} = b$, where $b \neq 0$ is a constant, and $w_0 = \epsilon$, where $\epsilon \ll 1$. Find the values for $w_1, \ldots, w_3$ and show that

$$w_4 = b + \mathcal{O}(\epsilon).$$

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(b) Given $w_{n-1} = b$, $w_n = \epsilon$, where $\epsilon \ll 1$, find w_{n+4} for the non-autonomous case. Can you find the condition that must be satisfied by z_n for this iterate to be bounded and analytic in b?

References

[1] C. Cresswell and N. Joshi, *The discrete first, second and thirty-fourth Painlevé hierarchies*, J. Phys. A: Math. Gen **32** (1999), 655–669.

[2] J. Hietarinta, N. Joshi, and F. W. Nijhoff, *Discrete Systems and Integrability*, Cambridge University Press, 2016.

[3] M. Noumi, *Painlevé Equations through Symmetry*, American Mathematical Society, 2004.

Tutorial 2

1. Consider $\mathbb{R}^3$, with basis given by unit vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. Show that the vectors
$$\alpha_1 = e_1 - e_2,$$
$$\alpha_2 = e_2 - e_3,$$
are simple roots forming the A_2 root system. Write down the corresponding expressions for the co-roots ${\alpha_1}^\vee$ and ${\alpha_2}^\vee$. Can you find the corresponding weights h_1 and h_2?
2. For constants $g_2, g_3 \in \mathbb{C}$, consider the Weierstrass cubics
$$f(x, y) = y^2 - 4x^3 + g_2\, x + g_3.$$
   (a) For the case $g_2 = 12$, find the value(s) of g_3 for which the curve $f(x, y) = 0$ is singular.
   (b) For generic g_2, find the relationship between g_2 and g_3 that must hold for the curve $f(x, y) = 0$ to be singular.
   (c) Resolve the curve in part (a) at its singular point.
   (d) Suppose $g_3 = 0$ and g_2 is free. Find the base points of the one-parameter family of curves $f(x, y) = 0$.
   (e) Assume $g_2 = 12$ but g_3 is not equal to any singular value found in part (a). Find the base points of the pencil $f(x, y) = 0$.
   (f) Find a good resolution of the pencil of curves for the case in part (e).
3. Find a good resolution of the pencil of curves
$$f(x, y) = x\, y^2 + x^2\, y - \beta\,(x + y) - \gamma\, x\, y = 0.$$