Vector equilibrium and asymptotics of zeros of multiple orthogonal polynomials

Andrei Martínez Finkelshtein
University of Almería

opsf s7
University of Kent, 2017

Multiple Orthogonal Polynomials

Number Theory

Rational approximation

Multiple Orthogonal Polynomials (MOP)

Interacting Particle Models

Random Matrix Theory

Integrable systems

I hope you all attended this course:

Course 3:

'Multiple Orthogonal Polynomials'

by **Walter Van Assche** (KU Leuven, Belgium)

Abstract. Multiple orthogonal polynomials are polynomials of one variable that satisfy orthogonality conditions with respect to r > 1 measures. They appeared as denominators of Hermite-Padé approximants to several functions in the 19th century and for that reason they are also known

HERMITE-PADÉ OR MULTIPLE O.P.

Hermite–Padé polynomials of type II: split the orthogonality conditions among several measures or weights.

Classical: $w_1, w_2 \geq 0$ on $\mathbb{R}$, and $P_{n,m}$ of degree $\leq N = n + m$:

$$\int x^j P_{n,m}(x) w_1(x) dx = 0, \quad j = 0, 1, \ldots, n-1,$$

$$\int x^j P_{n,m}(x) w_2(x) dx = 0, \quad j = 0, 1, \ldots, m-1.$$

Case 1: w_1 w_2

Angelesco or Angelescu

Case 2: w_1 w_2 + extra condition on w_2/w_1

Nikishin

Electrostatic toolbox for R

Stress level:

A FAMOUS FAMILY (OF POLYNOMIALS)

Who are these?

$$P_n^{(\alpha,\beta)}(z) = 2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}(z-1)^k(z+1)^{n-k}$$

Answer here:

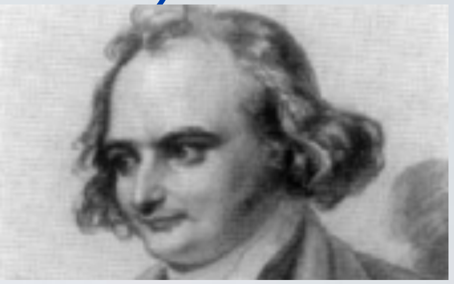
Course 1:

"Properties of Orthogonal Polynomials"

by **Kerstin Jordaan** (University of South Africa, South Africa)

Abstract. In these lectures, an introduction will be given to the theory of orthogonal polynomials. We discuss basic concepts and known properties of orthogonal polynomials within the context of applications. The lectures aim to show, by means of accessible examples, that interesting

A FAMOUS FAMILY (OF POLYNOMIALS)



Jacobi:

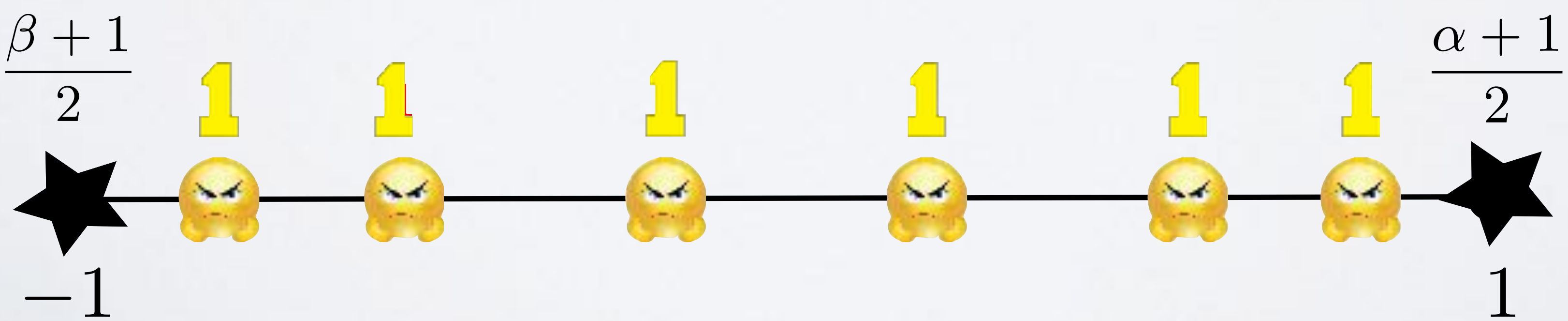
$$P_n^{(\alpha,\beta)}(z) = 2^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (z-1)^k (z+1)^{n-k}$$

For $\alpha, \beta > -1$ they form a well-known family of orthogonal polynomials on $[-1,1]$:

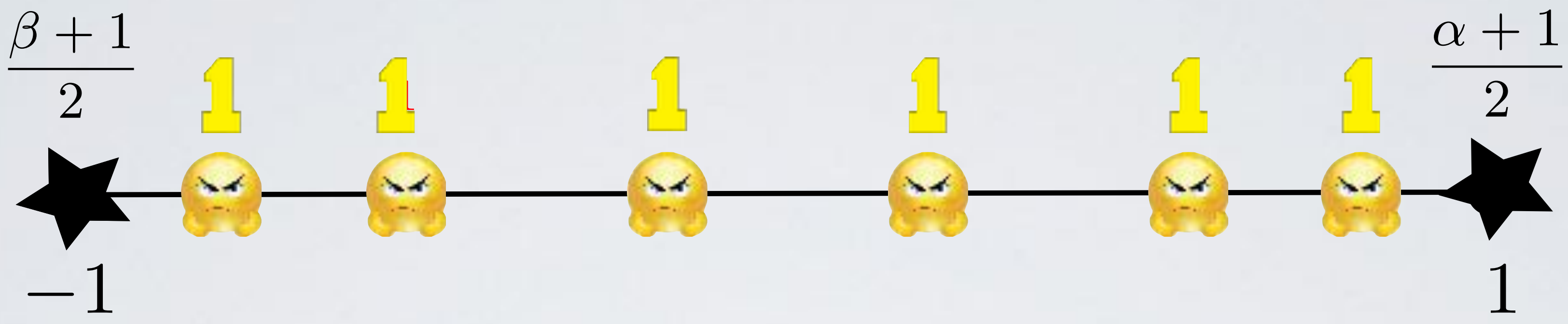
$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) x^k (1-x)^\alpha (1+x)^\beta dx = 0, \quad k = 0, 1, \ldots, n-1.$$

In consequence, all zeros of $P_n^{(\alpha,\beta)}$ are simple and lie on $(-1,1)$.

Stieltjes (1885) gave an electrostatic interpretation to these zeros:



A FAMOUS FAMILY (OF POLYNOMIALS)



Wild guess: in the $n \to \infty$ limit, the zeros should still follow a certain equilibrium distribution, minimizing a certain interaction energy.

Trivial observation: for $P(z) = (z - a_1)\dots(z - a_n)$,

$$-\log|P(z)|^{1/n} = \frac{1}{n}\sum_{j=1}^{n} \log\frac{1}{|z - a_j|} = \text{ logarithmic potential of } \nu(P)$$

Hence, in the $n \to \infty$ limit, $-\log|P_n|^{1/n}$ **should** look like the logarithmic potential of such an equilibrium distribution.

We need to develop a set of tools to make this guess precise and rigorous.

ELECTROSTATIC MODEL

a positive, signed or complex-valued measure on $\mathbb{C}$

$$\mu$$

logarithmic potential

$$V^{\mu}(z) := \int \log \frac{1}{|t-z|} d\mu(t)$$

Cauchy transform or m-function

$$C^{\mu}(z) := \int \frac{1}{t-z} d\mu(t)$$

mutual logarithmic energy

$$\langle \mu, \sigma \rangle := \iint \log \frac{1}{|t-z|} d\mu(t) d\sigma(z) = \int V^{\mu}(z) d\sigma(z)$$

logarithmic energy

$$I(\mu) := \langle \mu, \mu \rangle = \iint \log \frac{1}{|t-z|} d\mu(t) d\mu(z)$$

ELECTROSTATIC MODEL

logarithmic energy

$$I(\mu) := \langle \mu, \mu \rangle = \iint \log \frac{1}{|t-z|} d\mu(t) d\mu(z)$$

For $K \subset \mathbb{C}$ compact, the Robin constant is

$$\kappa = \min \{ I(\mu) : \mu \text{ unit measure on } K \}$$

The unique minimizer μ_K, such that $I(\mu_K) = \kappa$, is the equilibrium measure of K.

Value $\text{cap}(K) = e^{-\kappa}$ is the logarithmic capacity of K.

Also, $V^{\mu_K}(z) = \kappa - g_D(z, \infty)$, where $g(\cdot, K)$ is the Green function of $D = \mathbb{C} \setminus K$ with pole at ∞.

μ_K can be characterized by other extremal properties, such as

$$\max_{\mu \text{ on } K} \min_{z \in K} V^{\mu}(z)$$

ELECTROSTATIC MODEL

In the case of the "standard" (Hermitian) orthogonality,

$$\int_K \overline{Q_n(z)} z^k \, d\nu(z) = 0, \quad k = 0, 1, \ldots, n-1,$$

we have that this is equivalent to

$$\|Q_n\|^2_{L_2(\nu)} := \int_K |Q_n|^2 \, d\nu(z) = \min_{q(z)=z^n+\ldots} \|q\|^2_{L_2(\nu)}$$

We expect that extremality of $Q_n \Rightarrow$ extremality of their zero distribution: if ν is "sufficiently good", such that $\| \cdot \|_{L_2(\nu)} \sim \| \cdot \|_{L_\infty(\nu)}$, then

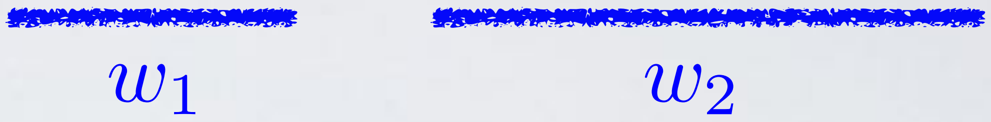
$$\|Q_n\|^{1/n}_{L_2(\nu)} \sim \exp\left(- \max_{\mu \text{ on } K} \min_{z \in K} V^\mu(z) \right)$$

which under some assumptions means that

$$\nu_n = \nu(Q_n) \xrightarrow{*} \mu_K$$

ELECTROSTATIC MODEL

Let us look at the possible electrostatic model for MOP in the simplest Angelesco case:



$$\int x^j P_{n,m}(x) w_1(x) dx = 0, \quad j = 0, 1, \ldots, n-1,$$

$$\int x^j P_{n,m}(x) w_2(x) dx = 0, \quad j = 0, 1, \ldots, m-1.$$

It is easy to prove that $P_{n,m}(z) = r_n(z) s_m(z)$, with all zeros of r_n and s_m in the right places.

$$\|P_{n,m}\|^2_{L_2(w_1)} = \int |r_n|^2 \, s_m(z) w_1(z) dz = \min_{r(z)=z^n+\ldots} \|r\|^2_{L_2(s_m w_1)}$$

$$\|P_{n,m}\|^2_{L_2(w_2)} = \int |s_m|^2 \, r_n(z) w_2(z) dz = \min_{s(z)=z^m+\ldots} \|s\|^2_{L_2(r_n w_2)}$$

Conclusion: all zeros of $P_{n,m}$ play a similar electrostatic role, but zeros on one of the interval count twice.

ELECTROSTATIC MODEL

for

vector-valued measures

$$\vec{\mu} = (\mu_1, \ldots, \mu_k)$$

vector of external fields

$$\vec{\varphi} = (\varphi_1, \ldots, \varphi_k)$$

$k \times k$ hermitian matrix of interactions

$$A \geq 0$$

mutual logarithmic energy

$$\langle \mu, \sigma \rangle := \iint \log \frac{1}{|t - z|} d\mu(t) d\sigma(z) = \int V^{\mu}(z) d\sigma(z)$$

the total energy

$$E(\vec{\mu}, \vec{\varphi}) = \langle \vec{\mu}, A\vec{\mu} \rangle + 2 \sum_{j=1}^{k} \int \varphi_j \, d\mu_j$$

ELECTROSTATIC MODEL

for

vector-valued measures

$$\vec{\mu} = (\mu_1, \ldots, \mu_k)$$

vector of external fields

$$\vec{\varphi} = (\varphi_1, \ldots, \varphi_k)$$

$k \times k$ hermitian matrix of interactions

$$A \geq 0$$

Angelesco case:

w_1 w_2

$k = 2$

$$A = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

the total energy

$$E(\vec{\mu}, \vec{\varphi}) = \langle \vec{\mu}, A\vec{\mu} \rangle + 2 \sum_{j=1}^{k} \int \varphi_j \, d\mu_j$$

ELECTROSTATIC MODEL

for

vector-valued measures

$$\vec{\mu} = (\mu_1, \ldots, \mu_k)$$

vector of external fields

$$\vec{\varphi} = (\varphi_1, \ldots, \varphi_k)$$

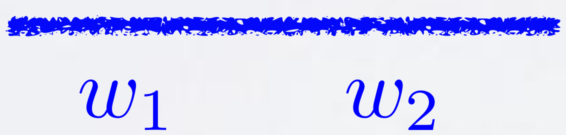
$k \times k$ hermitian matrix of interactions

$$A \geq 0$$

$k = 2$

$$A = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$$

Nikishin case:



the total energy

$$E(\vec{\mu}, \vec{\varphi}) = \langle \vec{\mu}, A\vec{\mu} \rangle + 2 \sum_{j=1}^{k} \int \varphi_j \, d\mu_j$$

CONCLUSION

$$E(\vec{\mu},\vec{\varphi}) = \langle \vec{\mu}, A\vec{\mu}\rangle + 2\sum_{j=1}^{k}\int \varphi_j \, d\mu_j \longrightarrow \min$$

The limit zero distribution of the MOP is given by a combination of the components of the vector measure solving the extremal problem (vector equilibrium measure) under additional constraints.

For instance, if

$$\int x^j P_{n,m}(x) w_1(x)dx = 0, \quad j = 0,1,\dots,n-1,$$

$$\int x^j P_{n,m}(x) w_2(x)dx = 0, \quad j = 0,1,\dots,m-1.$$

Easy?

and $n/(m+n) \to \alpha$ as $m,n\to\infty$, then we minimize E among all $\vec{\mu} = (\mu_1,\mu_2)$, with

$$|\mu_1| = \int d\mu_1 = \alpha, \quad |\mu_2| = \int d\mu_2 = 1-\alpha$$

CONCLUSION

$$E(\vec{\mu},\vec{\varphi}) = \langle \vec{\mu}, A\vec{\mu}\rangle + 2\sum_{j=1}^{k}\int \varphi_j \, d\mu_j \longrightarrow \min$$

Well, actually solving this problem is usually highly non-trivial, since you don't know a priori each component's support.

For instance, if

$$\int x^j P_{n,m}(x) w_1(x)dx = 0, \quad j = 0, 1, \ldots, n-1,$$

$$\int x^j P_{n,m}(x) w_2(x)dx = 0, \quad j = 0, 1, \ldots, m-1.$$

Easy?

and $n/(m+n) \to \alpha$ as $m, n \to \infty$, then we minimize E among all $\vec{\mu} = (\mu_1, \mu_2)$, with

$$|\mu_1| = \int d\mu_1 = \alpha, \quad |\mu_2| = \int d\mu_2 = 1-\alpha$$

Electrostatics on the plane

Stress level:

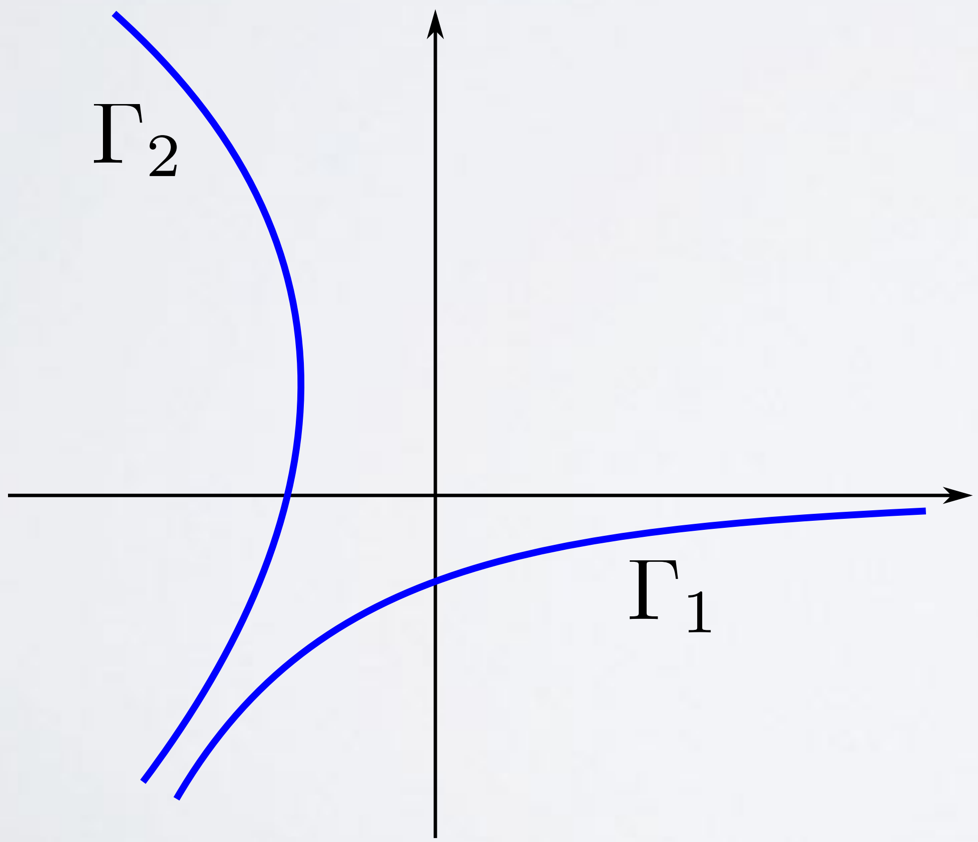
HERMITE-PADÉ FOR CUBIC WEIGHT

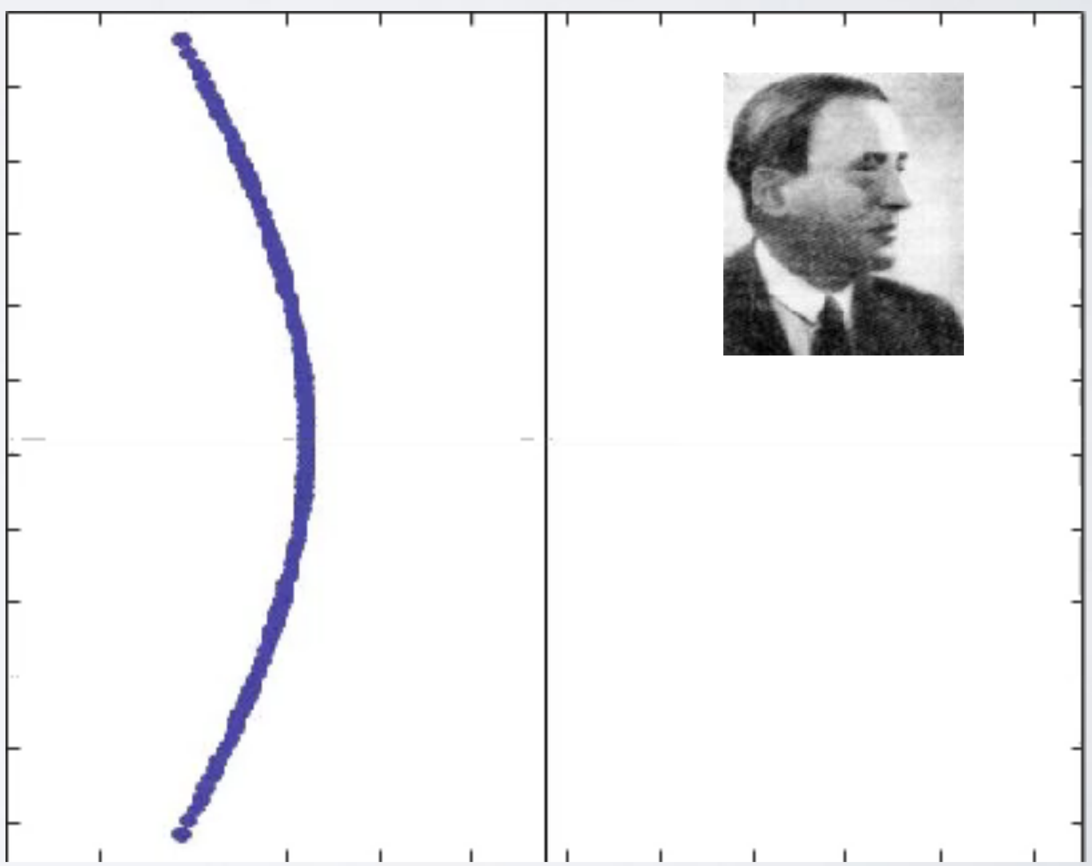
For $m, n, N \in \mathbb{N}$, $m+n = N$, we have a polynomial $P_{n,m}$ of degree $\leq N$ such that

$$\int_{\Gamma_1} z^j P_{n,m}(z) e^{-Nz^3} dz = 0, \quad j = 0, 1, \ldots, n-1,$$

$$\int_{\Gamma_2} z^j P_{n,m}(z) e^{-Nz^3} dz = 0, \quad j = 0, 1, \ldots, m-1$$

where





$N = 150$, n from 0 to 75

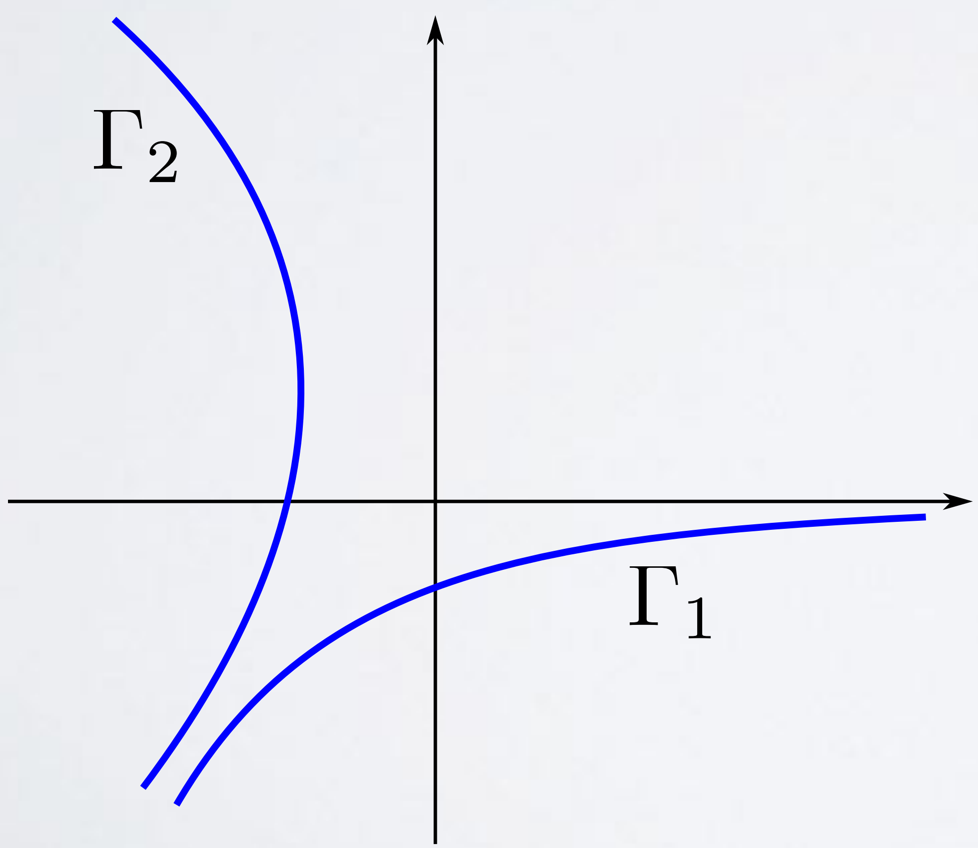
HERMITE-PADÉ FOR CUBIC WEIGHT

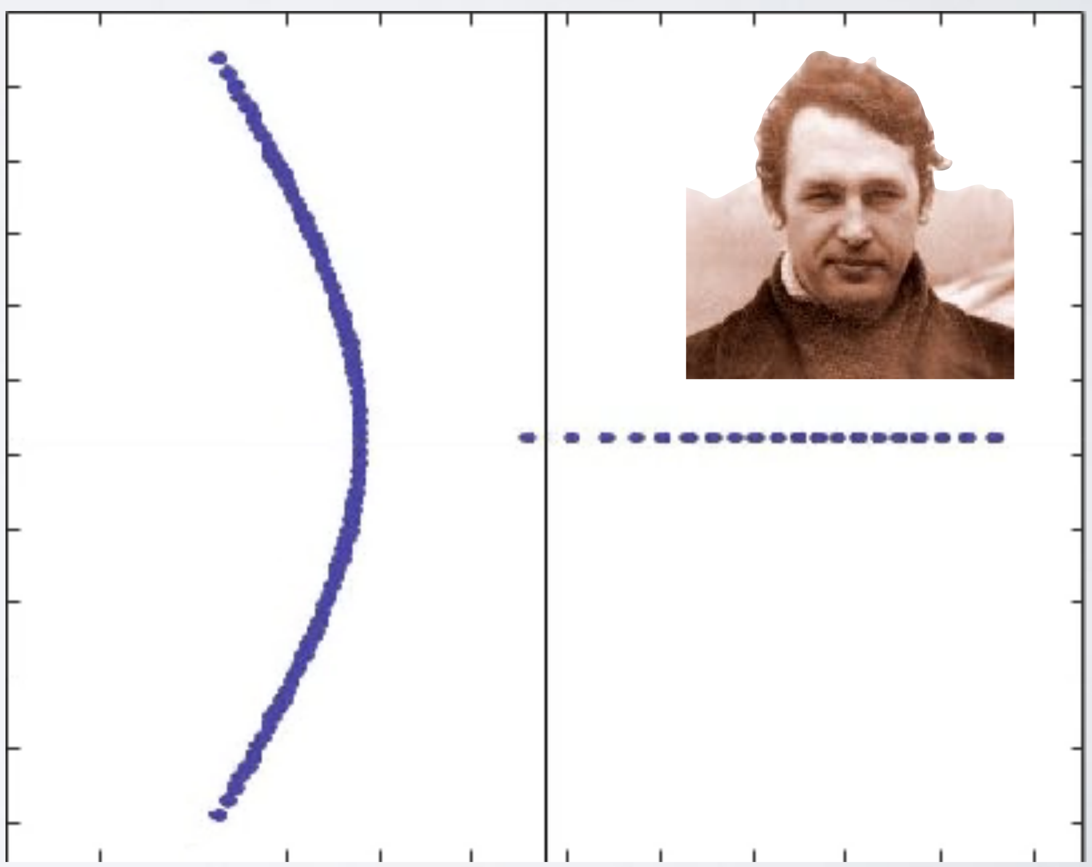
For $m, n, N \in \mathbb{N}$, $m+n = N$, we have a polynomial $P_{n,m}$ of degree $\leq N$ such that

$$\int_{\Gamma_1} z^j P_{n,m}(z) e^{-Nz^3} dz = 0, \quad j = 0, 1, \ldots, n-1,$$

$$\int_{\Gamma_2} z^j P_{n,m}(z) e^{-Nz^3} dz = 0, \quad j = 0, 1, \ldots, m-1$$

where





$N = 150$, n from 0 to 75

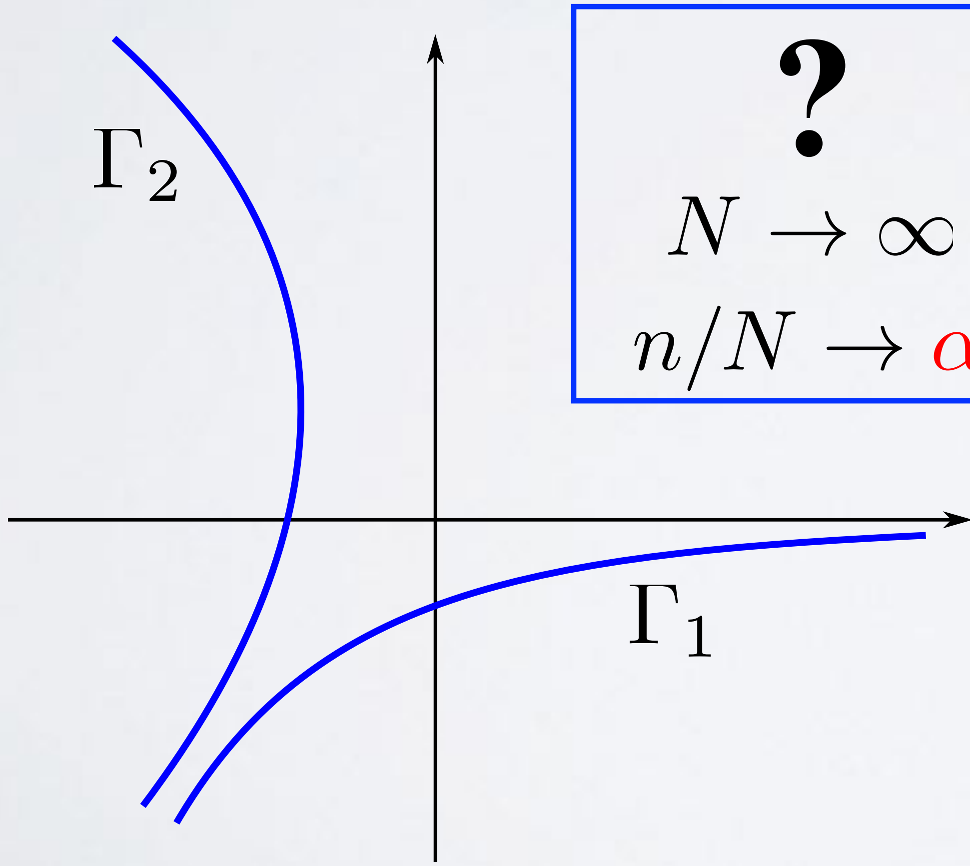
HERMITE-PADÉ FOR CUBIC WEIGHT

For $m, n, N \in \mathbb{N}$, $m+n = N$, we have a polynomial $P_{n,m}$ of degree $\leq N$ such that

$$\int_{\Gamma_1} z^j P_{n,m}(z) e^{-Nz^3} dz = 0, \quad j = 0, 1, \ldots, n-1,$$

$$\int_{\Gamma_2} z^j P_{n,m}(z) e^{-Nz^3} dz = 0, \quad j = 0, 1, \ldots, m-1$$

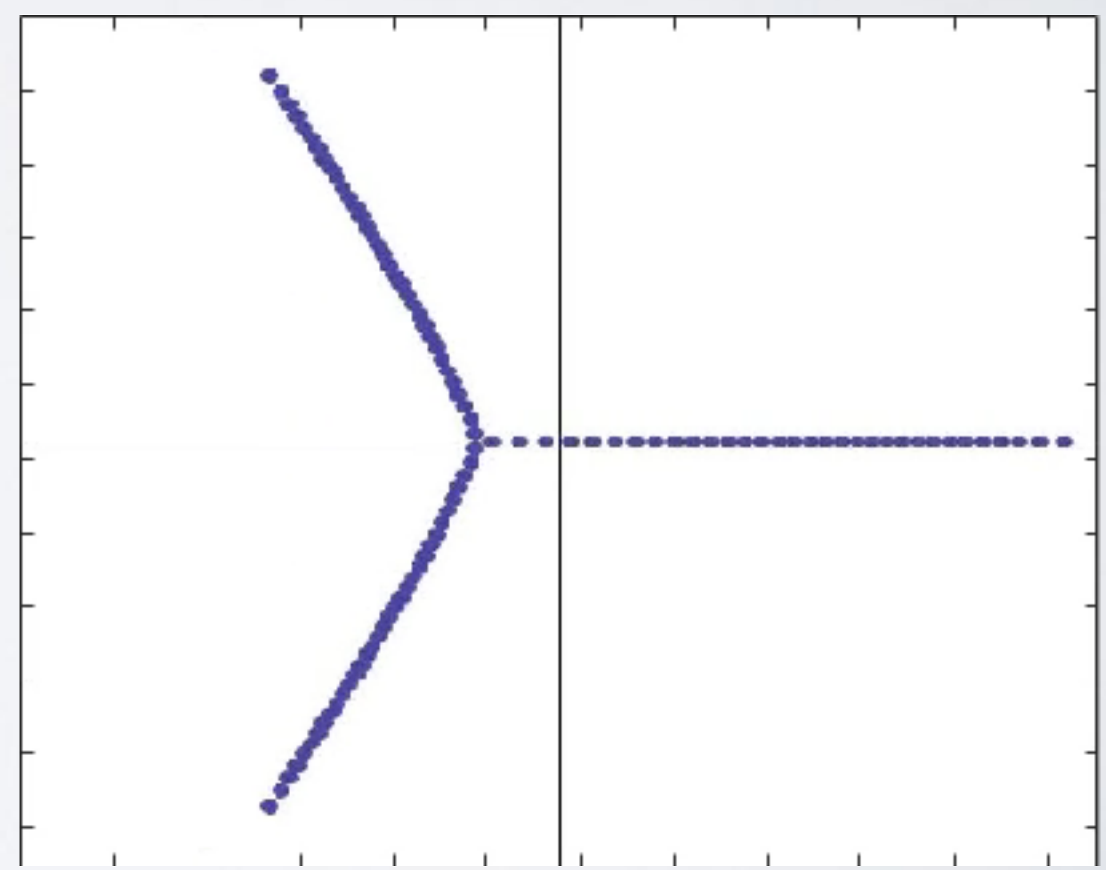
where



?

$N \to \infty$

$n/N \to \alpha$



$N = 150$, n from 0 to 75

NON-HERMTIAN ORTHOGONALITY

$$\int_{\text{curve}} (\text{analytic function})(z)dz = 0$$

Electrostatics:

For the non-hermitian orthogonality, the whole complex plane is a conductor.

The logarithmic energy has no global minima sobre $\mathbb{C}$.

The role of the equilibrium measures for $\mathbb{R}$ is played by solutions of the max-min type problems

$$\max_{\Gamma} \min_{\mu \text{ on } \Gamma} I(\mu)$$

The solutions, called critical measures are saddle points of the energy.

VECTOR CRITICAL MEASURES

vector-valued measures

$$\vec{\mu} = (\mu_1, \mu_2, \mu_3)$$

vector of external fields

$$\vec{\varphi} = (\mathrm{Re}(z^3), \mathrm{Re}(z^3), 0)$$

matrix of interactions

$$A = (a_{j,k}) = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & -1/2 \\ 1/2 & -1/2 & 1 \end{pmatrix},$$

the total energy

$$E(\vec{\mu}, \vec{\varphi}) = \langle \vec{\mu}, A\vec{\mu} \rangle + 2 \int \mathrm{Re}(z^3)\, d(\mu_1 + \mu_2)$$

Additional conditions on $\vec{\mu}$:

$$|\mu_1| + |\mu_2| = 1, \quad |\mu_1| + |\mu_3| = \alpha, \quad |\mu_2| - |\mu_3| = 1 - \alpha$$

VECTOR CRITICAL MEASURES

vector-valued measures

$$\vec{\mu} = (\mu_1, \mu_2, \mu_3)$$

vector of external fields

$$\vec{\varphi} = (\mathrm{Re}(z^3), \mathrm{Re}(z^3), 0)$$

Accumulated knowledge:

The (normalized) zero-counting measure of $P_{n,m}$'s converges, as $n, m \to \infty$, $n/(n+m) \to \alpha$, to $\mu_1 + \mu_2$, where $\vec{\mu}$ is a saddle point of $E(\vec{\mu}, \vec{\varphi})$ on the plane.

Vector critical measures

the total energy

$$E(\vec{\mu}, \vec{\varphi}) = \langle \vec{\mu}, A\vec{\mu} \rangle + 2 \int \mathrm{Re}(z^3)\, d(\mu_1 + \mu_2)$$

Additional conditions on $\vec{\mu}$:

$$|\mu_1| + |\mu_2| = 1, \quad |\mu_1| + |\mu_3| = \alpha, \quad |\mu_2| - |\mu_3| = 1 - \alpha$$

The hunting of the ~~Spark~~ vector critical measures

Welcome to the ride, please hold tight

Stress level:

VECTOR CRITICAL MEASURES

Critical measures have a feature: the Cauchy transform of their components are related to the same algebraic equation!

Recall that for a measure μ we have denoted

$$C^{\mu}(z) = \int \frac{d\mu(t)}{t-z}$$

AMF-Silva, 2015: if $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$ is a vector critical measure, then

$$\begin{aligned}
\xi_1(z) &= 2z^2 + C^{\mu_1}(z) + C^{\mu_2}(z)\\
\xi_2(z) &= -z^2 - C^{\mu_1}(z) - C^{\mu_3}(z)\\
\xi_3(z) &= -z^2 - C^{\mu_2}(z) + C^{\mu_3}(z)
\end{aligned}$$

are the three solutions of

$$\boxed{\xi^3 - R(z)\xi + D(z) = 0}$$

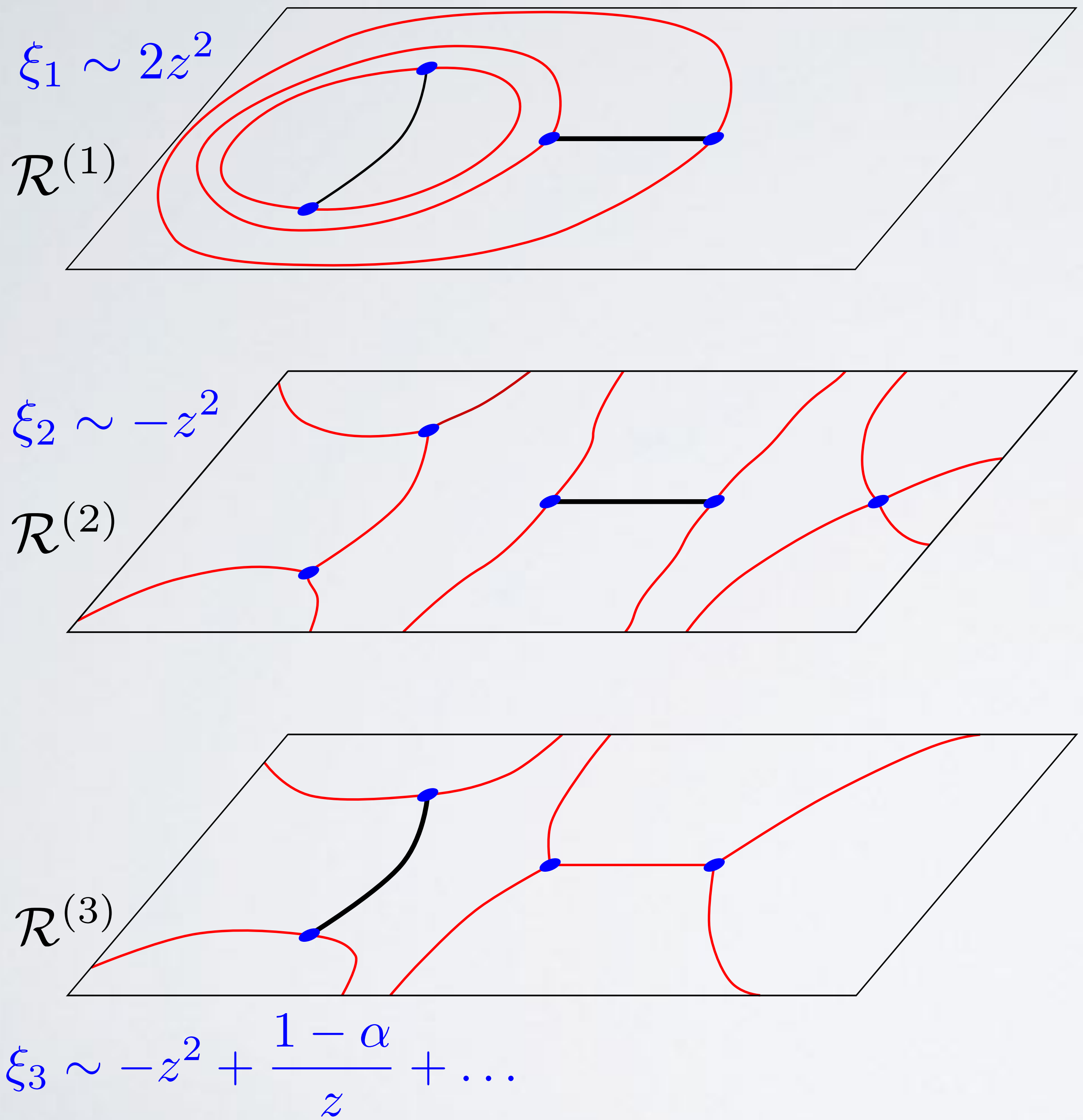
with

$$R(z) = 3z^4 - 3z - c, \quad D(z) = -2z^6 + 3z^3 + cz^2 - 3\alpha(1-\alpha),$$

where $c = c(\alpha)$ is explicit.

VECTOR CRITICAL MEASURES

$\xi^3 - R(z)\xi + D(z) = 0 \quad \Longrightarrow \quad$ a 3-sheeted Riemann surface $\mathcal{R}$ of genus 0, with 4 branch points and 1 node.



Facts:

$$Q(z) := \begin{cases} (\xi_2 - \xi_3)^2 & \text{on } \mathcal{R}_1, \\ (\xi_1 - \xi_3)^2 & \text{on } \mathcal{R}_2, \\ (\xi_2 - \xi_1)^2 & \text{on } \mathcal{R}_3 \end{cases}$$

is globally defined and meromorphic on $\mathcal{R}$

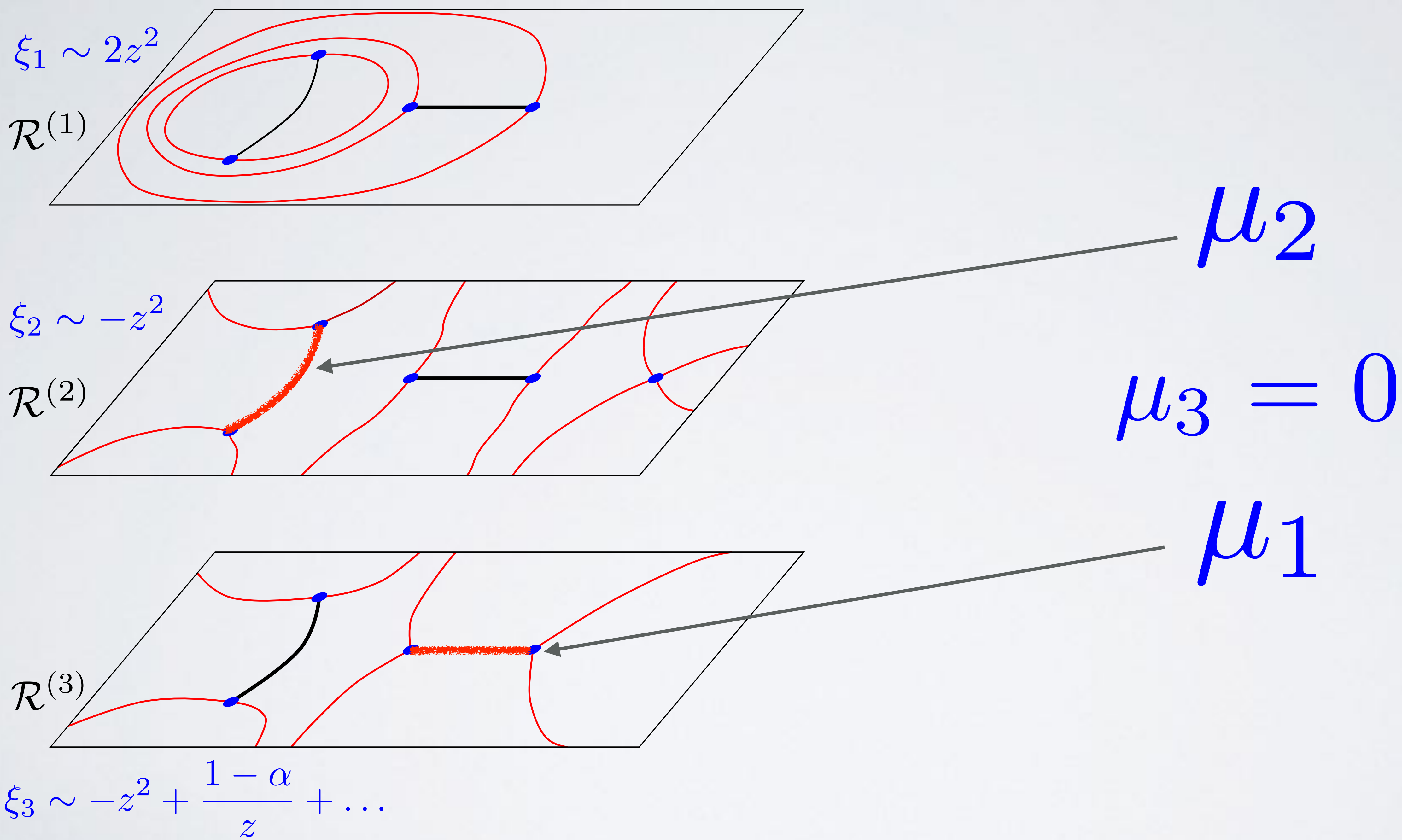
Components of $\vec{\mu}$ live on trajectories of the quadratic differential $Q(z)dz^2$ on $\mathcal{R}$, i.e. on curves where

$$\operatorname{Re} \int^z \sqrt{Q(t)}\,dt \equiv \text{const}$$

For small values of α we can relatively easily identify the trajectories whose projections on $\mathbb{C}$ carry $\vec{\mu}$

VECTOR CRITICAL MEASURES

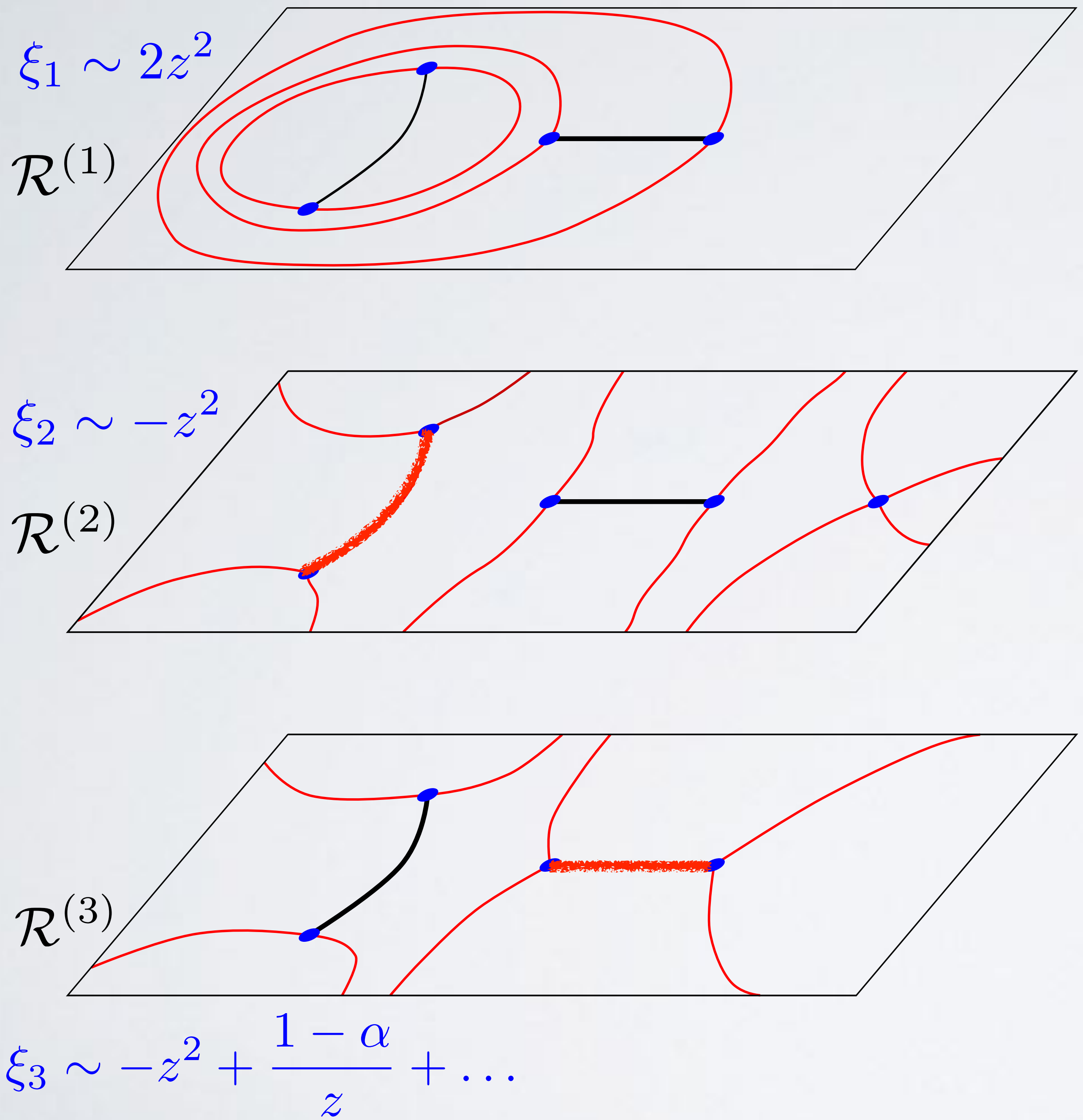
$\xi^3 - R(z)\xi + D(z) = 0 \quad \Longrightarrow$ a 3-sheeted Riemann surface $\mathcal{R}$ of genus 0, with 4 branch points and 1 node.



For small values of α we can relatively easily identify the trajectories whose projections on $\mathbb{C}$ carry $\vec{\mu}$

VECTOR CRITICAL MEASURES

$\xi^3 - R(z)\xi + D(z) = 0 \quad \Longrightarrow \quad$ a 3-sheeted Riemann surface $\mathcal{R}$ of genus 0, with 4 branch points and 1 node.



For the rest of the values of α we have to trace all the deformations of these trajectories on $\mathcal{R}$ and their phase transitions

This task could be matter of an independent talk, with trajectories of quadratic differentials as main characters!

For small values of α we can relatively easily identify the trajectories whose projections on $\mathbb{C}$ carry $\vec{\mu}$

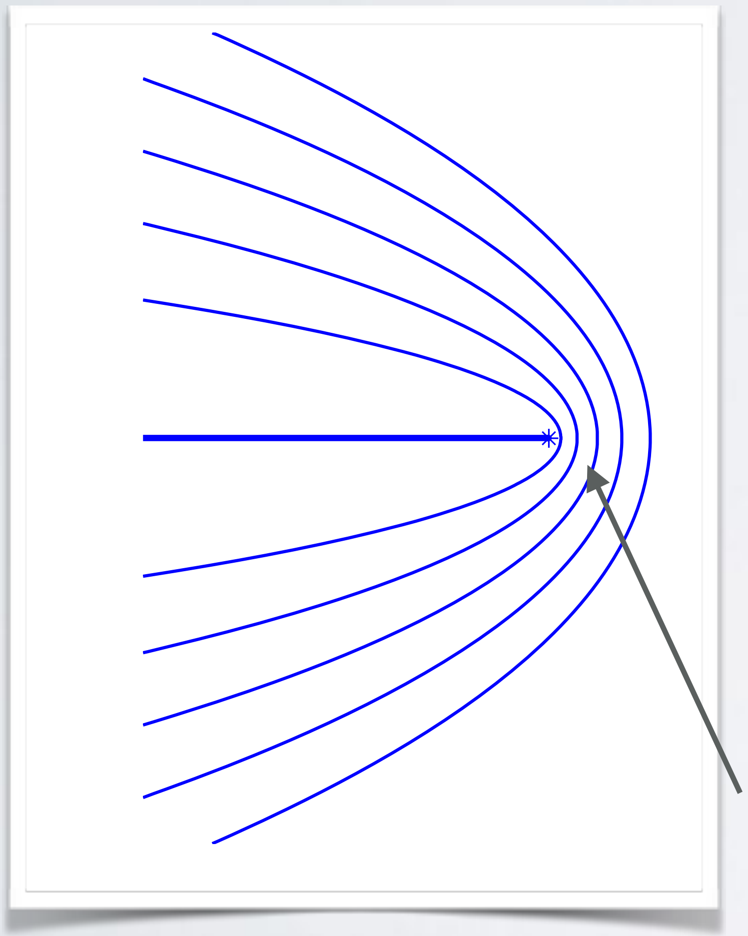
Intermezzo: trajectories of quadratic differentials

Stress level:

TRAJECTORIES OF QUADRATIC DIFFERENTIALS

A (vertical) trajectory arc of a quadratic differential (q.d.) $\varpi = Q(z)dz^2$ in a local parameter z on a Riemann surface $\mathcal{R}$ is a curve $\gamma : (a, b) \to \mathcal{R}$ s.t.

$$\operatorname{Re} \int_\gamma \sqrt{Q(z)}dz \equiv \text{const}$$

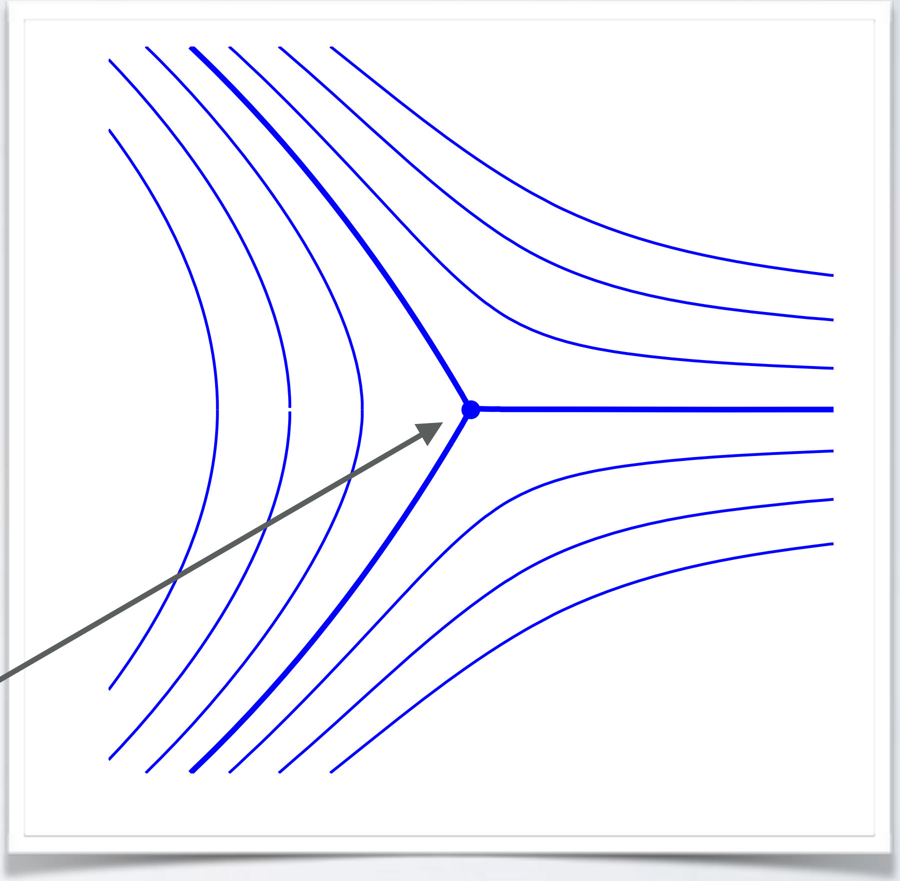


Simple pole

It is an integral curve of

$$Q(z) \left(\frac{dz}{dt}\right)^2 = -1$$

Simple zero



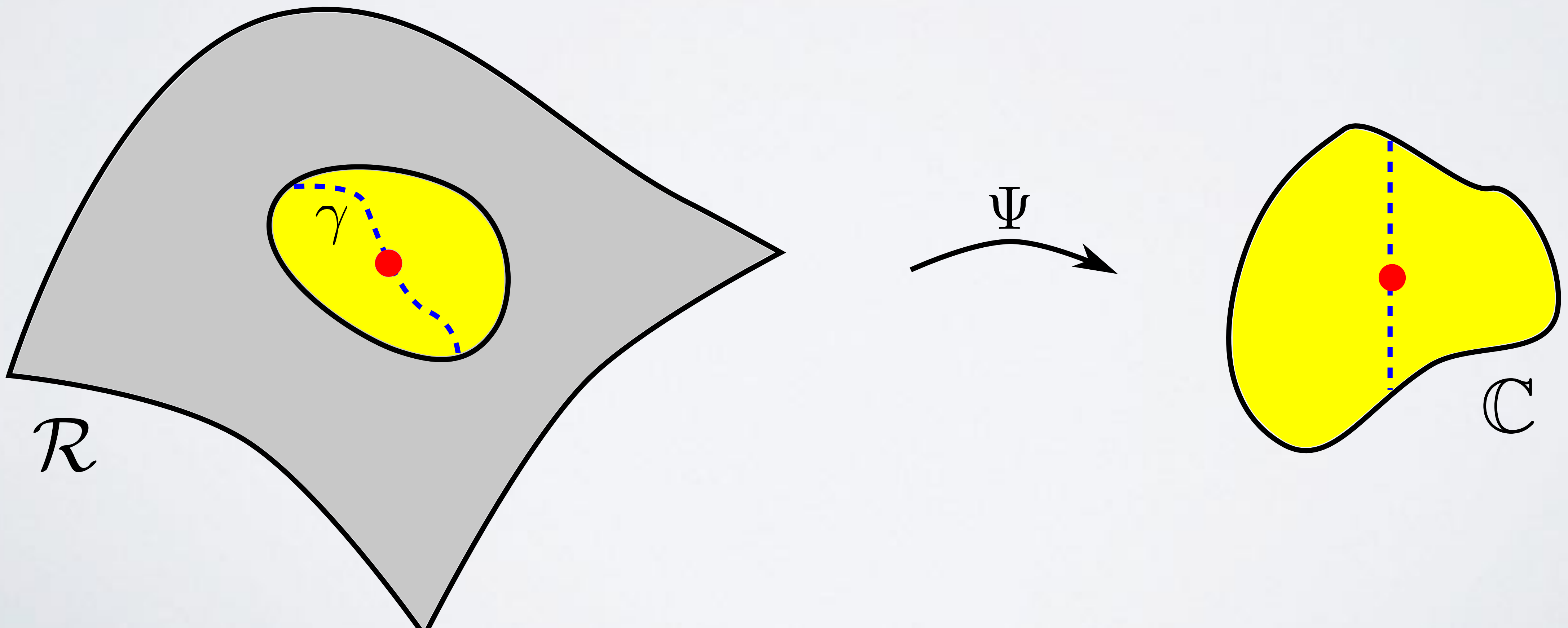
TRAJECTORIES OF QUADRATIC DIFFERENTIALS

A (vertical) trajectory arc of a quadratic differential (q.d.) $\varpi = Q(z)dz^2$ in a local parameter z on a Riemann surface $\mathcal{R}$ is a curve $\gamma : (a,b) \to \mathcal{R}$ s.t.

$$\mathrm{Re} \int_\gamma \sqrt{Q(z)}dz \equiv \mathrm{const}$$

At each regular point of the q.d. we can define the map

$$\Psi(z) := \int^z \sqrt{Q(z)}dz$$



TRAJECTORIES OF QUADRATIC DIFFERENTIALS

A (vertical) trajectory arc of a quadratic differential (q.d.) $\varpi = Q(z)dz^2$ in a local parameter z on a Riemann surface $\mathcal{R}$ is a curve $\gamma : (a,b) \to \mathcal{R}$ s.t.

$$\mathrm{Re} \int_\gamma \sqrt{Q(z)}dz \equiv \text{const}$$

At each regular point of the q.d. we can define the map

$$\Psi(z) := \int^z \sqrt{Q(z)}dz$$

The global structure of the trajectories of a q.d. can be very complicated: trajectories might be closed, critical (i.e. joining a pair of zeros or poles of the q.d.) or even recurrent or dense in a domain.

$$\mathcal{G} := \bigcup \text{critical trajectories} = \text{the critical graph of } \varpi$$

Theorem: *$\mathcal{R} \setminus \mathcal{G}$ is a finite union of canonical domains on $\mathcal{R}$.*

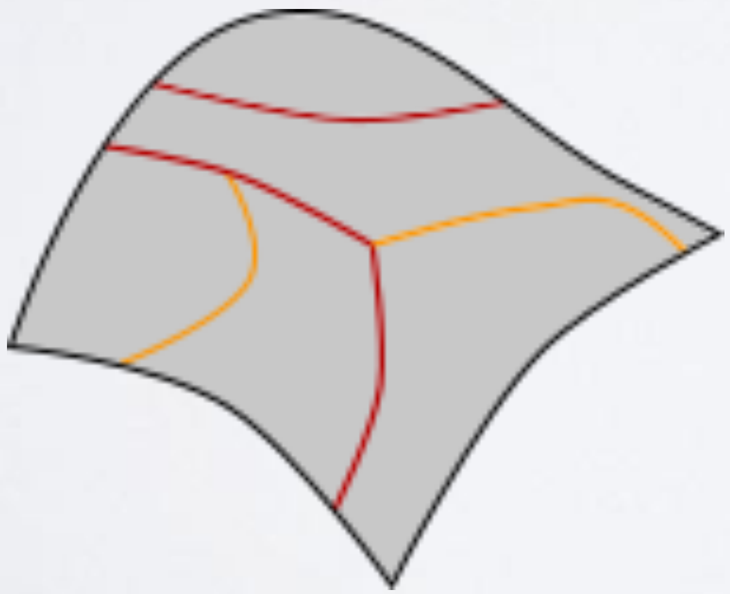
TRAJECTORIES OF QUADRATIC DIFFERENTIALS

A (vertical) trajectory arc of a quadratic differential (q.d.) $\varpi = Q(z)dz^2$ in a local parameter z on a Riemann surface $\mathcal{R}$ is a curve $\gamma : (a,b) \to \mathcal{R}$ s.t.

$$\mathrm{Re} \int_\gamma \sqrt{Q(z)}dz \equiv \mathrm{const}$$

At each regular point of the q.d. we can define the map

$$\Psi(z) := \int^z \sqrt{Q(z)}dz$$



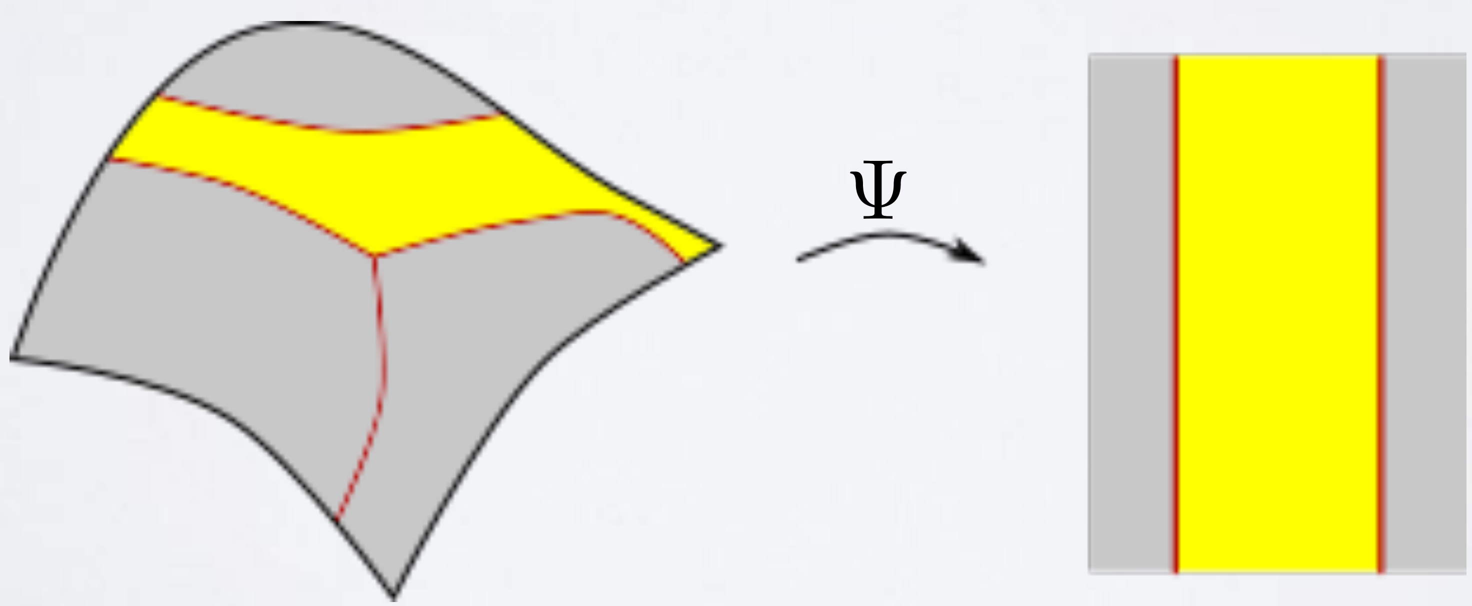
TRAJECTORIES OF QUADRATIC DIFFERENTIALS

A (vertical) trajectory arc of a quadratic differential (q.d.) $\varpi = Q(z)dz^2$ in a local parameter z on a Riemann surface $\mathcal{R}$ is a curve $\gamma : (a,b) \to \mathcal{R}$ s.t.

$$\mathrm{Re} \int_\gamma \sqrt{Q(z)}dz \equiv \mathrm{const}$$

At each regular point of the q.d. we can define the map

$$\Psi(z) := \int^z \sqrt{Q(z)}dz$$



Strip domain

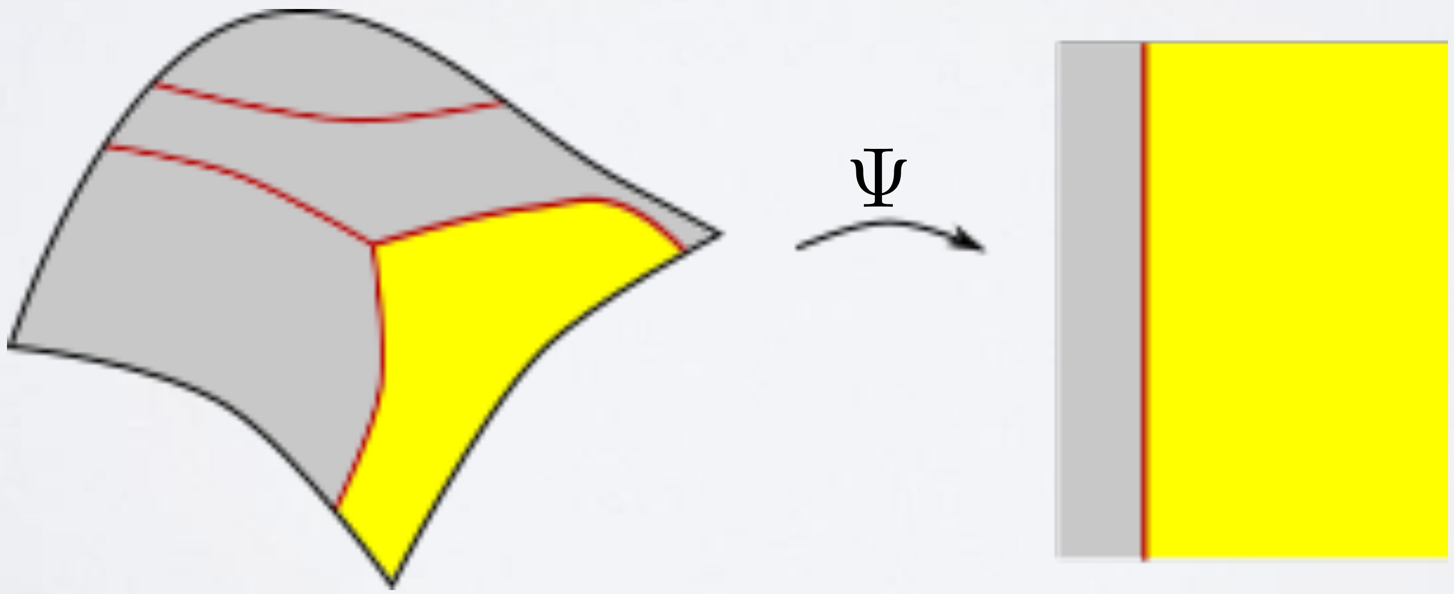
TRAJECTORIES OF QUADRATIC DIFFERENTIALS

A (vertical) trajectory arc of a quadratic differential (q.d.) $\varpi = Q(z)dz^2$ in a local parameter z on a Riemann surface $\mathcal{R}$ is a curve $\gamma : (a, b) \to \mathcal{R}$ s.t.

$$\operatorname{Re} \int_\gamma \sqrt{Q(z)}dz \equiv \text{const}$$

At each regular point of the q.d. we can define the map

$$\Psi(z) := \int^z \sqrt{Q(z)}dz$$



Half-plane domain

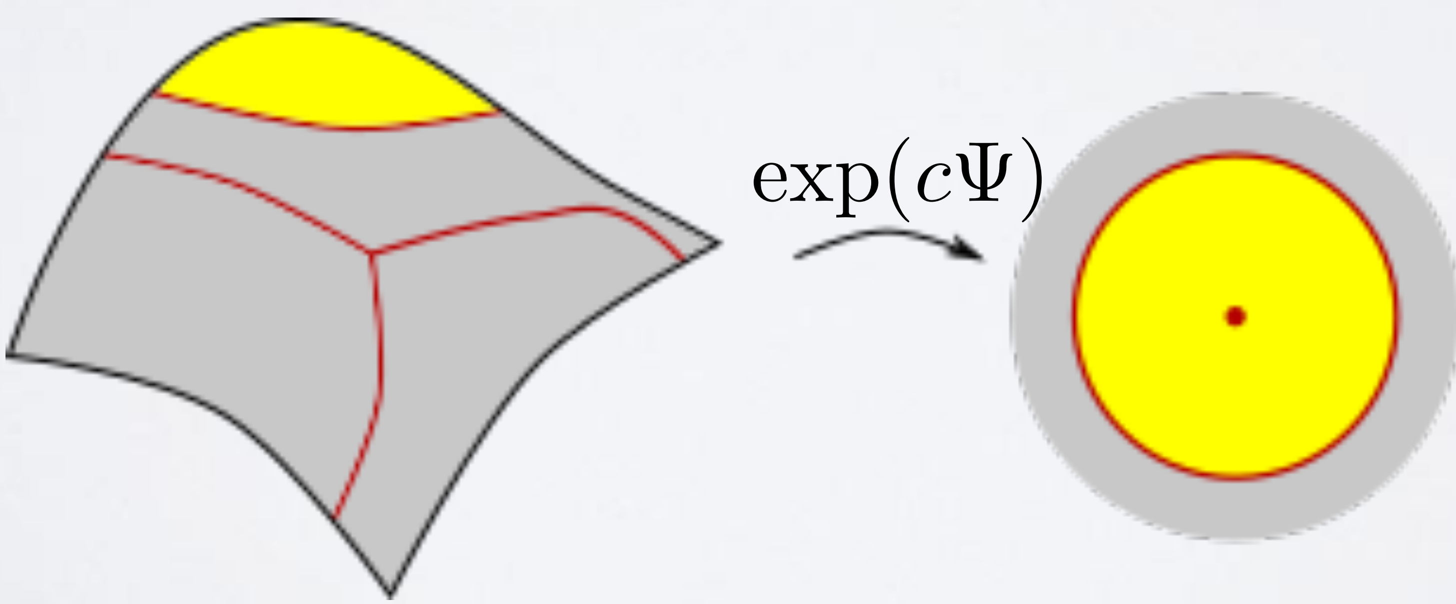
TRAJECTORIES OF QUADRATIC DIFFERENTIALS

A (vertical) trajectory arc of a quadratic differential (q.d.) $\varpi = Q(z)dz^2$ in a local parameter z on a Riemann surface $\mathcal{R}$ is a curve $\gamma : (a, b) \to \mathcal{R}$ s.t.

$$\mathrm{Re} \int_\gamma \sqrt{Q(z)}dz \equiv \mathrm{const}$$

At each regular point of the q.d. we can define the map

$$\Psi(z) := \int^z \sqrt{Q(z)}dz$$



Circle domain

TRAJECTORIES OF QUADRATIC DIFFERENTIALS

A (vertical) trajectory arc of a quadratic differential (q.d.) $\varpi = Q(z)dz^2$ in a local parameter z on a Riemann surface $\mathcal{R}$ is a curve $\gamma : (a,b) \to \mathcal{R}$ s.t.

$$\mathrm{Re} \int_\gamma \sqrt{Q(z)}dz \equiv \mathrm{const}$$

At each regular point of the q.d. we can define the map

$$\Psi(z) := \int^z \sqrt{Q(z)}dz$$

If p and q lie on the boundary of a canonical domain, we can use

$$\sigma := \int_p^q \sqrt{Q(z)}dz$$

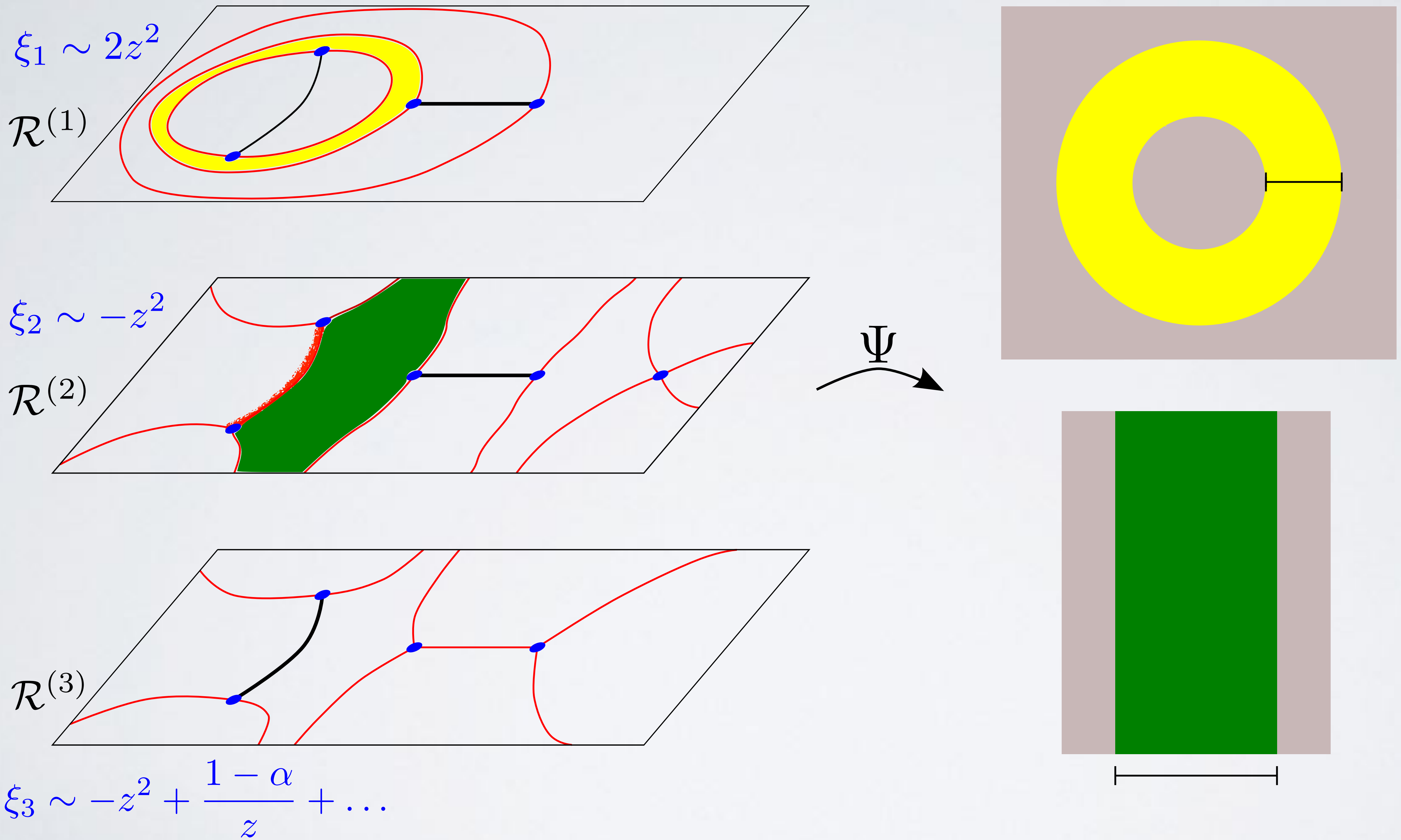
as parameters that control the deformation of the critical graph. These parameters (or "widths") are related to moduli of families of curves on $\mathcal{R}$.

Back to our Riemann surface

Stress level:

VECTOR CRITICAL MEASURES

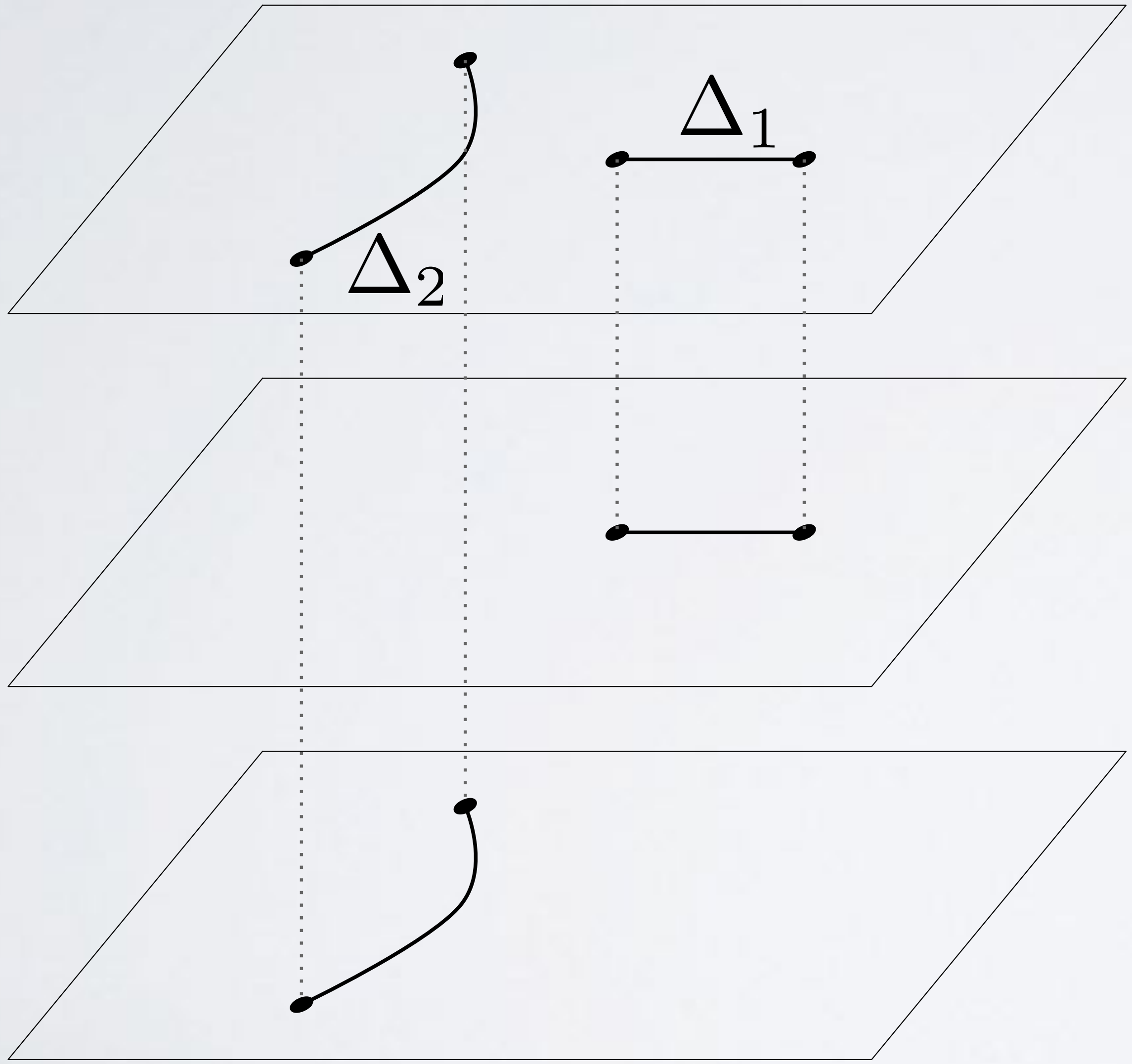
$\xi^3 - R(z)\xi + D(z) = 0 \implies$ a 3-sheeted Riemann surface $\mathcal{R}$ of genus 0, with 4 branch points and 1 node.

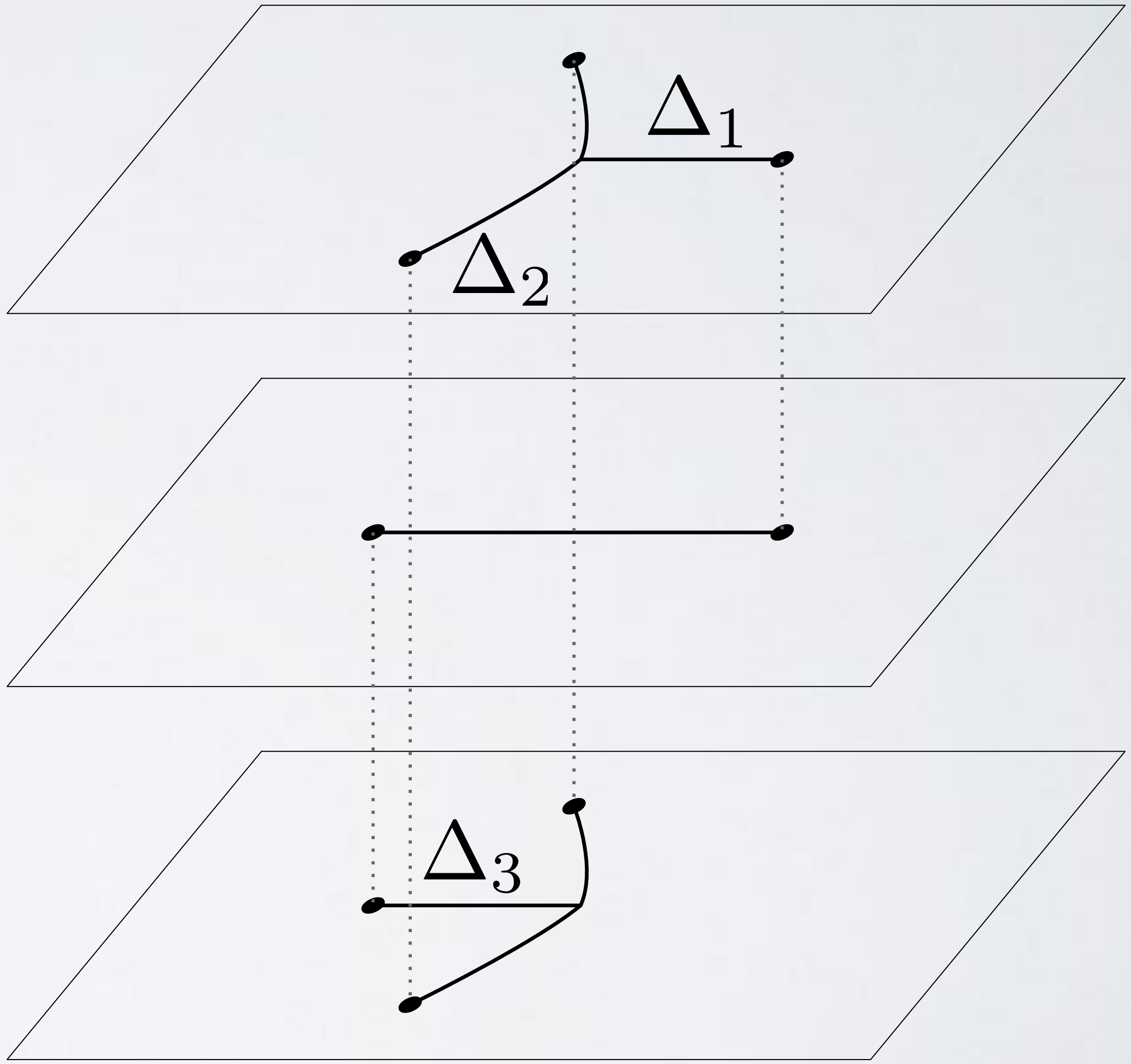


As α varies, we can keep track of the structure of the critical graph using our parameters or "widths", σ.

VECTOR CRITICAL MEASURES

This leaves us with the following "pre-critical" (left) and "post-critical" sheet structure of $\mathcal{R}$:

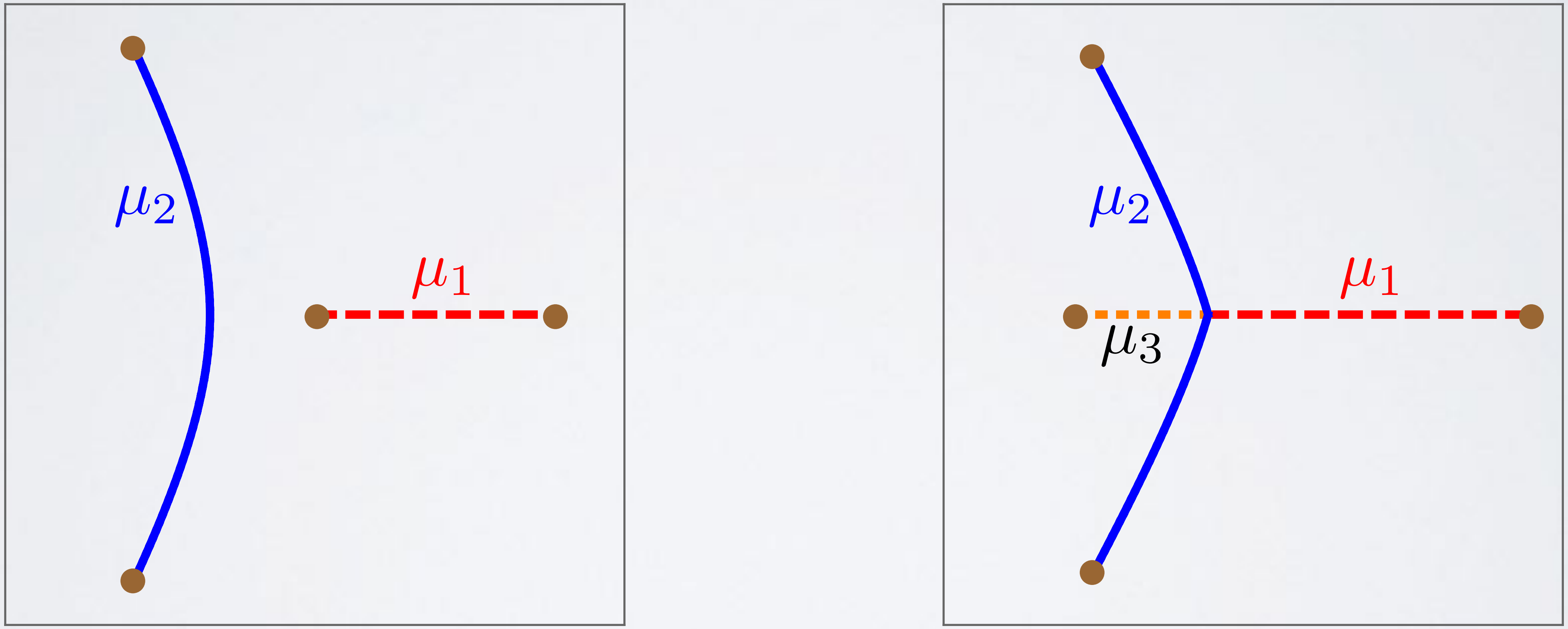




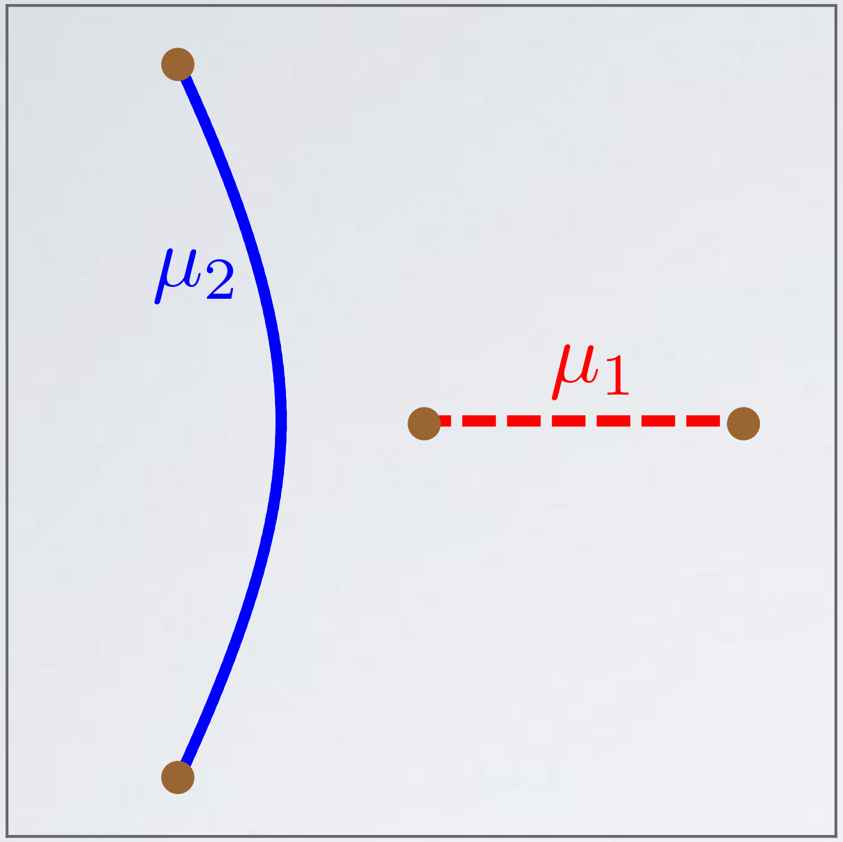
VECTOR CRITICAL MEASURES

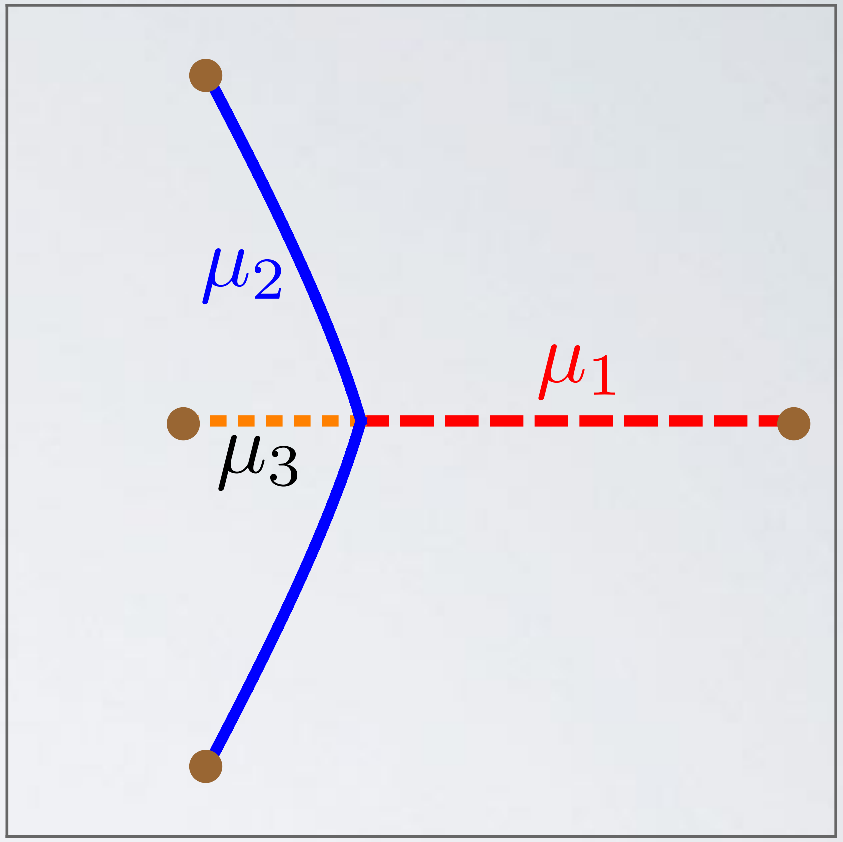
This leaves us with the following "pre-critical" (left) and "post-critical" sheet structure of $\mathcal{R}$:

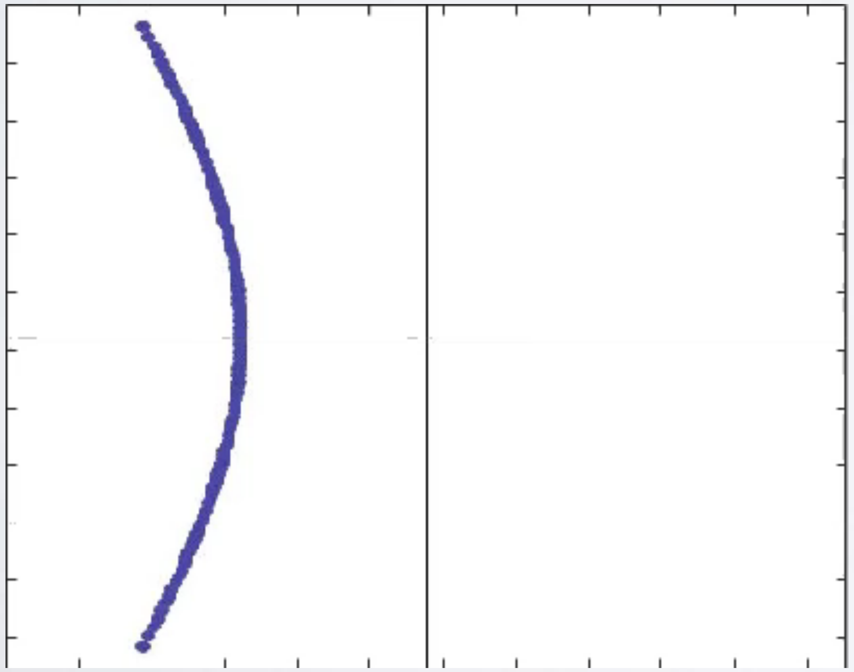
Which in turn gives us the following structure of the support of the vector critical measure:



WEAK ASYMPTOTICS





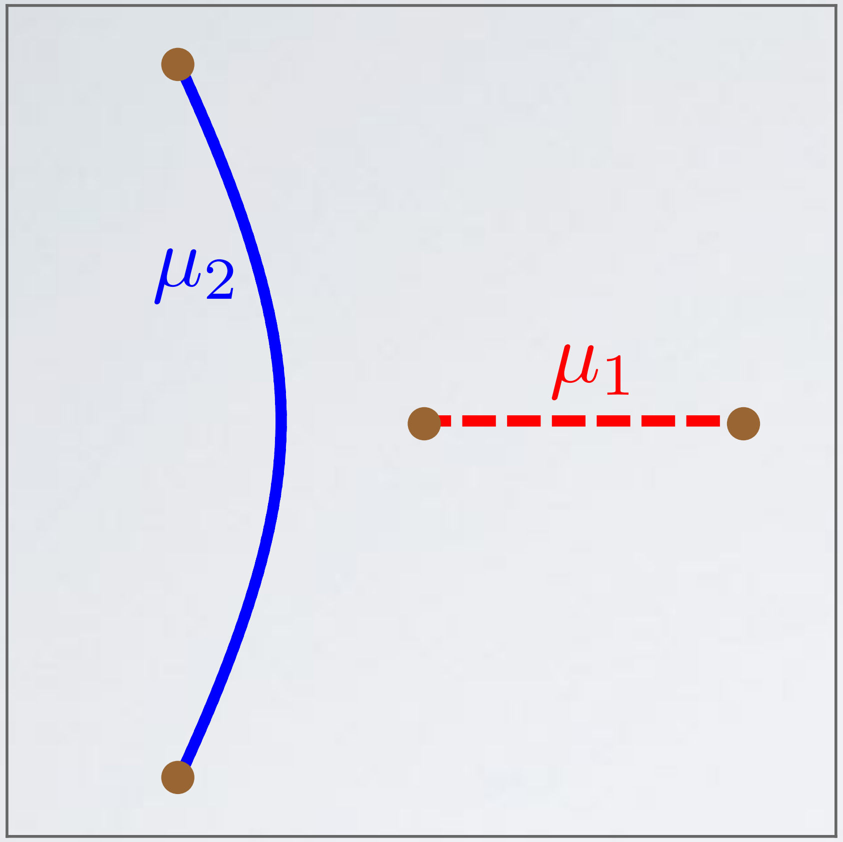


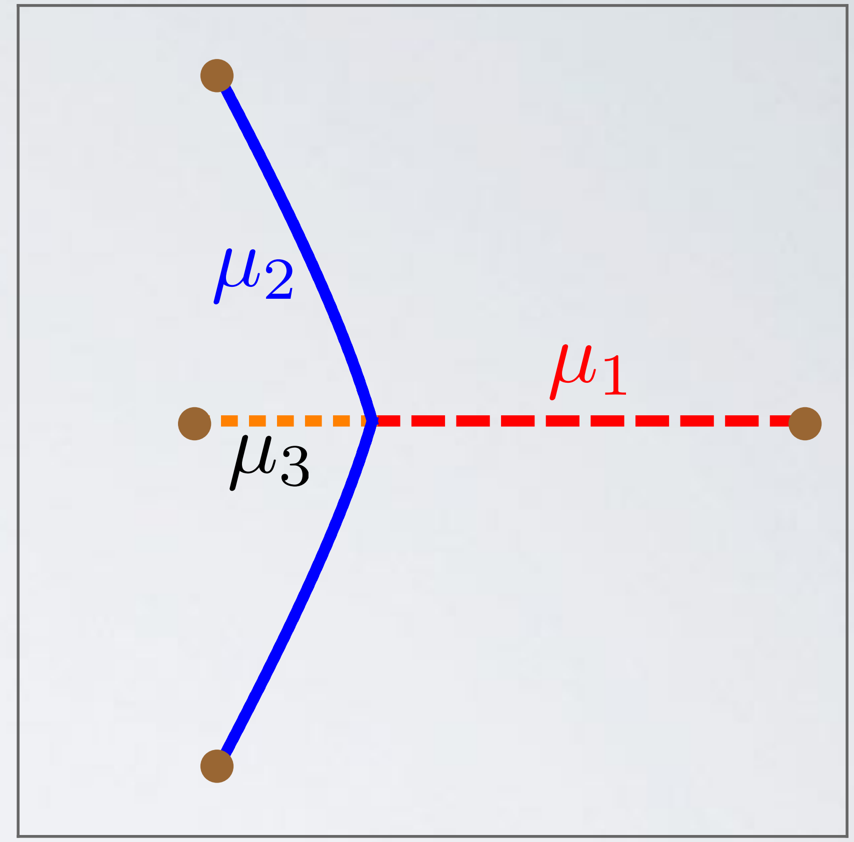
$$\frac{1}{n+m} \sum_{P_{m,n}(z)=0} \delta_z \to \mu_1 + \mu_2$$

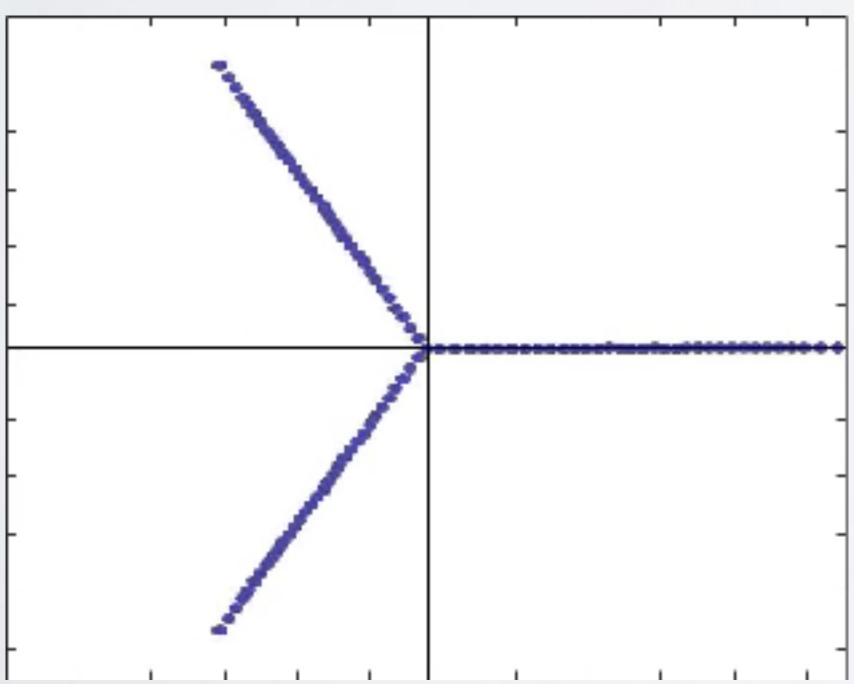
$$\text{as } n, m \to \infty, \quad \frac{n}{n+m} \to \alpha$$

μ_j's have explicit expressions in terms of ξ_j's

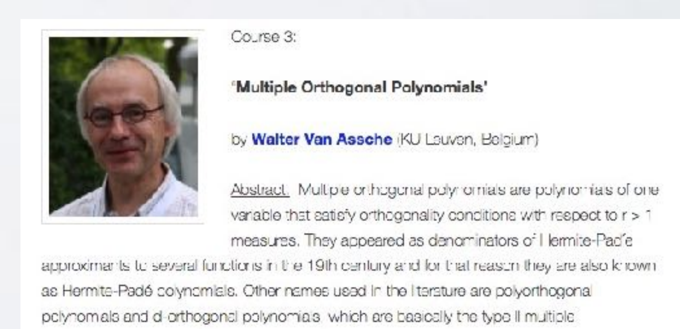
WEAK ASYMPTOTICS







In fact, the vector critical measures are also a key ingredient for the Riemann-Hilbert asymptotic analysis of these Hermite-Padé polynomials.



Course 3:

'Multiple Orthogonal Polynomials'

by **Walter Van Assche** (KU Leuven, Belgium)

Abstract. Multiple orthogonal polynomials are polynomials of one variable that satisfy orthogonality conditions with respect to r > 1 measures. They appeared as denominators of Hermite-Padé approximants to several functions in the 19th century and for that reason they are also known as Hermite-Padé polynomials. Other names used in the literature are polyorthogonal polynomials and d-orthogonal polynomials which are basically the type II multiple

Thank you