

Multiple Orthogonal Polynomials

Walter Van Assche

Summer school on OPSF, University of Kent

26–30 June, 2017

Plan of the course

Plan of the course

lecture 1: Definitions + basic properties

Plan of the course

lecture 1: Definitions + basic properties

lecture 2: Hermite-Padé, Multiple Hermite polynomials

Plan of the course

lecture 1: Definitions + basic properties

lecture 2: Hermite-Padé, Multiple Hermite polynomials

lecture 3: Multiple Laguerre polynomials (first and second kind)

Plan of the course

lecture 1: Definitions + basic properties

lecture 2: Hermite-Padé, Multiple Hermite polynomials

lecture 3: Multiple Laguerre polynomials (first and second kind)

lecture 4: Multiple Jacobi polynomials:
Jacobi-Angelesco + Jacobi-Piñeiro

Plan of the course

lecture 1: Definitions + basic properties

lecture 2: Hermite-Padé, Multiple Hermite polynomials

lecture 3: Multiple Laguerre polynomials (first and second kind)

lecture 4: Multiple Jacobi polynomials:
Jacobi-Angelesco + Jacobi-Piñeiro

lecture 5: Riemann-Hilbert problem

Multiple Laguerre polynomials

The Laguerre weight is

$$w(x) = x^\alpha e^{-x}, \quad x \in [0, \infty), \quad \alpha > -1.$$

Multiple Laguerre polynomials

The Laguerre weight is

$$w(x) = x^\alpha e^{-x}, \quad x \in [0, \infty), \quad \alpha > -1.$$

There are two easy ways to obtain multiple Laguerre polynomials:

Multiple Laguerre polynomials

The Laguerre weight is

$$w(x) = x^\alpha e^{-x}, \quad x \in [0, \infty), \quad \alpha > -1.$$

There are two easy ways to obtain multiple Laguerre polynomials:

- 1 Changing the parameter α to $\alpha_1, \dots, \alpha_r$

This gives **multiple Laguerre polynomials of the first kind**

Multiple Laguerre polynomials

The Laguerre weight is

$$w(x) = x^\alpha e^{-x}, \quad x \in [0, \infty), \quad \alpha > -1.$$

There are two easy ways to obtain multiple Laguerre polynomials:

- 1 Changing the parameter α to $\alpha_1, \dots, \alpha_r$
This gives **multiple Laguerre polynomials of the first kind**
- 2 Changing the exponential decay at infinity from e^{-x} to $e^{-c_j x}$
with parameters c_1, \dots, c_r
This gives **multiple Laguerre polynomials of the second kind**

Multiple Laguerre I

Type II multiple Laguerre of the first kind: $L_{\vec{n}}^{\vec{\alpha}}(x)$

$$\int_0^{\infty} x^k L_{\vec{n}}^{\vec{\alpha}}(x) x^{\alpha_j} e^{-x} dx = 0, \quad 0 \leq k \leq n_j - 1,$$

for $1 \leq j \leq r$.

Multiple Laguerre I

Type II multiple Laguerre of the first kind: $L_{\vec{n}}^{\vec{\alpha}}(x)$

$$\int_0^{\infty} x^k L_{\vec{n}}^{\vec{\alpha}}(x) x^{\alpha_j} e^{-x} dx = 0, \quad 0 \leq k \leq n_j - 1,$$

for $1 \leq j \leq r$.

Parameters $\alpha_j > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$. (AT system)

Multiple Laguerre I

Type II multiple Laguerre of the first kind: $L_{\vec{n}}^{\vec{\alpha}}(x)$

$$\int_0^{\infty} x^k L_{\vec{n}}^{\vec{\alpha}}(x) x^{\alpha_j} e^{-x} dx = 0, \quad 0 \leq k \leq n_j - 1,$$

for $1 \leq j \leq r$.

Parameters $\alpha_j > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$. (AT system)

Rodrigues formula:

$$(-1)^{|\vec{n}|} e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x) = \prod_{j=1}^r \left(x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} x^{n_j + \alpha_j} \right) e^{-x}.$$

Explicit formula:

$$L_{\vec{n}}^{\vec{\alpha}}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} (-1)^{|\vec{k}|} \frac{n_1!}{(n_1 - k_1)!} \cdots \frac{n_r!}{(n_r - k_r)!} \\ \times \binom{n_r + \alpha_r}{k_r} \binom{n_r + n_{r-1} + \alpha_{r-1} - k_r}{k_{r-1}} \cdots \binom{|\vec{n}| - |\vec{k}| + k_1 + \alpha_1}{k_1} x^{|\vec{n}| - |\vec{k}|}$$

Explicit formula:

$$L_{\vec{n}}^{\vec{\alpha}}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} (-1)^{|\vec{k}|} \frac{n_1!}{(n_1 - k_1)!} \cdots \frac{n_r!}{(n_r - k_r)!} \\ \times \binom{n_r + \alpha_r}{k_r} \binom{n_r + n_{r-1} + \alpha_{r-1} - k_r}{k_{r-1}} \cdots \binom{|\vec{n}| - |\vec{k}| + k_1 + \alpha_1}{k_1} x^{|\vec{n}| - |\vec{k}|}$$

$$(-1)^{|\vec{n}|} e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x) = \prod_{j=1}^r (\alpha_j + 1)_{n_j} {}_rF_r \left(\begin{matrix} n_1 + \alpha_1 + 1, \dots, n_r + \alpha_r + 1 \\ \alpha_1 + 1, \dots, \alpha_r + 1 \end{matrix} \middle| -x \right).$$

Recurrence relation:

$$xL_{\vec{n}}(x) = L_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k}L_{\vec{n}}(x) + \sum_{j=1}^r a_{\vec{n},j}L_{\vec{n}-\vec{e}_j}(x)$$

$$a_{\vec{n},j} = n_j(n_j + \alpha_j) \prod_{i=1, i \neq j}^r \frac{n_j + \alpha_j - \alpha_i}{n_j - n_i + \alpha_j - \alpha_i},$$

$$b_{\vec{n},k} = |\vec{n}| + n_k + \alpha_k + 1.$$

Multiple Laguerre I: differential properties

Raising operators:

$$\frac{d}{dx} \left(x^{\alpha_j+1} e^{-x} L_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j}(x) \right) = -x^{\alpha_j} e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x), \quad 1 \leq j \leq r.$$

Multiple Laguerre I: differential properties

Raising operators:

$$\frac{d}{dx} \left(x^{\alpha_j+1} e^{-x} L_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j}(x) \right) = -x^{\alpha_j} e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x), \quad 1 \leq j \leq r.$$

Lowering operator:

$$\frac{d}{dx} L_{\vec{n}}^{\vec{\alpha}}(x) = \sum_{j=1}^r \frac{\prod_{i=1}^r (n_i + \alpha_i - \alpha_j)}{\prod_{i=1, i \neq j}^r (\alpha_i - \alpha_j)} L_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j}(x).$$

Multiple Laguerre I: differential properties

Raising operators:

$$\frac{d}{dx} \left(x^{\alpha_j+1} e^{-x} L_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j}(x) \right) = -x^{\alpha_j} e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x), \quad 1 \leq j \leq r.$$

Lowering operator:

$$\frac{d}{dx} L_{\vec{n}}^{\vec{\alpha}}(x) = \sum_{j=1}^r \frac{\prod_{i=1}^r (n_i + \alpha_i - \alpha_j)}{\prod_{i=1, i \neq j}^r (\alpha_i - \alpha_j)} L_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j}(x).$$

Differential equation:

$$\left(\prod_{j=1}^r D_j \right) D L_{\vec{n}}^{\vec{\alpha}}(x) = - \sum_{j=1}^r \frac{\prod_{i=1}^r (n_i + \alpha_i - \alpha_j)}{\prod_{i=1, i \neq j}^r (\alpha_i - \alpha_j)} \left(\prod_{i \neq j} D_i \right) L_{\vec{n}}^{\vec{\alpha}}(x).$$

$$D = \frac{d}{dx}, \quad D_j = x^{-\alpha_j} e^x D x^{\alpha_j+1} e^{-x}.$$

Multiple Laguerre I: integral representations

Integral representations:

$$e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x) = \frac{(-1)^{|\vec{n}|}}{2\pi i} \oint_{\Sigma} x^{-z-1} \Gamma(z+1) \prod_{j=1}^r (\alpha_j - z)_{n_j} dz$$

Σ is a Hankel contour from $-\infty - i0+$, around 0, to $-\infty + i0+$.

Multiple Laguerre I: integral representations

Integral representations:

$$e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x) = \frac{(-1)^{|\vec{n}|}}{2\pi i} \oint_{\Sigma} x^{-z-1} \Gamma(z+1) \prod_{j=1}^r (\alpha_j - z)_{n_j} dz$$

Σ is a Hankel contour from $-\infty - i0+$, around 0, to $-\infty + i0+$.

$$(-1)^{|\vec{n}|} Q_{\vec{n}}(x) = \frac{e^{-x}}{2\pi i} \oint_{\Gamma} \frac{x^z}{\Gamma(z+1) \prod_{j=1}^r (\alpha_j - z)_{n_j}} dz$$

Γ is a contour enclosing $\alpha_j, \alpha_j + 1, \dots, \alpha_j + n_j - 1$ for $1 \leq j \leq r$.

Multiple Laguerre I: integral representations

Integral representations:

$$e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x) = \frac{(-1)^{|\vec{n}|}}{2\pi i} \oint_{\Sigma} x^{-z-1} \Gamma(z+1) \prod_{j=1}^r (\alpha_j - z)_{n_j} dz$$

Σ is a Hankel contour from $-\infty - i0+$, around 0, to $-\infty + i0+$.

$$(-1)^{|\vec{n}|} Q_{\vec{n}}(x) = \frac{e^{-x}}{2\pi i} \oint_{\Gamma} \frac{x^z}{\Gamma(z+1) \prod_{j=1}^r (\alpha_j - z)_{n_j}} dz$$

Γ is a contour enclosing $\alpha_j, \alpha_j + 1, \dots, \alpha_j + n_j - 1$ for $1 \leq j \leq r$.

$$\begin{aligned} (-1)^{|\vec{n}|} e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x) &= \frac{n_1! \cdots n_r!}{(2\pi i)^r} \\ &\times \oint_{\Sigma} \cdots \oint_{\Sigma} \exp\left(-\frac{x}{t_1 \cdots t_r}\right) \frac{t_1^{-\alpha_1-1} \cdots t_r^{-\alpha_r-1}}{(1-t_1)^{n_1+1} \cdots (1-t_r)^{n_r+1}} dt_1 \cdots dt_r \end{aligned}$$

Multiple Laguerre I: zero distribution

Theorem (E+J Coussement, VA)

Let $0 < x_{1,2n} < x_{2,2n} < \dots < x_{2n,2n}$ be the zeros of $L_{n,n}^{(\alpha_1, \alpha_2)}$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{k=1}^{2n} f\left(\frac{x_{k,2n}}{2n}\right) = \int_0^{27/4} f(x) w_2(x) dx,$$

for every continuous function f on $[0, 27/4]$, where $w_2(x) = 4/27 g(4x/27)$ with

$$g(x) = \frac{2\sqrt{3}}{16\pi} \frac{(1 + 3\sqrt{1-x})(1 - \sqrt{1-x})^{1/3} - (1 - 3\sqrt{1-x})(1 + \sqrt{1-x})^{1/3}}{x^{2/3}}.$$

Multiple Laguerre I: zero distribution

The proof uses the four-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) + d_n p_{n-2}(x),$$

$$p_{2n}(x) = L_{n,n}(x), \quad p_{2n+1}(x) = L_{n+1,n}(x)$$

where

$$b_{2n} = 3n + \alpha_1 + 1, \quad b_{2n+1} = 3n + \alpha_2 + 2,$$

$$c_{2n} = n(3n + \alpha_1 + \alpha_2), \quad c_{2n+1} = 3n^2 + n(\alpha_1 + \alpha_2 + 3) + \alpha_1 + 1,$$

$$d_{2n} = n(n + \alpha_1)(n + \alpha_1 - \alpha_2), \quad d_{2n+1} = n(n + \alpha_2)(n + \alpha_2 - \alpha_1).$$

Multiple Laguerre I: zero distribution

Theorem (Neuschel, VA)

Let $0 < x_{1,rn} < x_{2,rn} < \cdots < x_{rn,rn}$ be the zeros of $L_{n,n,\dots,n}^{\vec{\alpha}}$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{rn} \sum_{k=1}^{rn} f\left(\frac{x_{k,rn}}{rn}\right) = \int_0^{c_r} f(x/r) u_r(x) dx, \quad c_r = \frac{(r+1)^{r+1}}{r^r},$$

where u_r is given by

$$u_r(x) = \frac{1}{r\pi} \frac{(\sin r\varphi)^{r+1}}{(\sin(r+1)\varphi)^r},$$

where

$$x = \frac{(\sin(r+1)\varphi)^{r+1}}{\sin\varphi(\sin r\varphi)^r}, \quad 0 < \varphi < \frac{\pi}{r+1}.$$

Multiple Laguerre polynomials: zero distribution

Theorem (Van Assche)

Let $\vec{n} = (\lfloor q_1 n \rfloor, \dots, \lfloor q_r n \rfloor)$, where $q_j > 0$ ($1 \leq j \leq r$) and $q_1 + \dots + q_r = 1$. Suppose

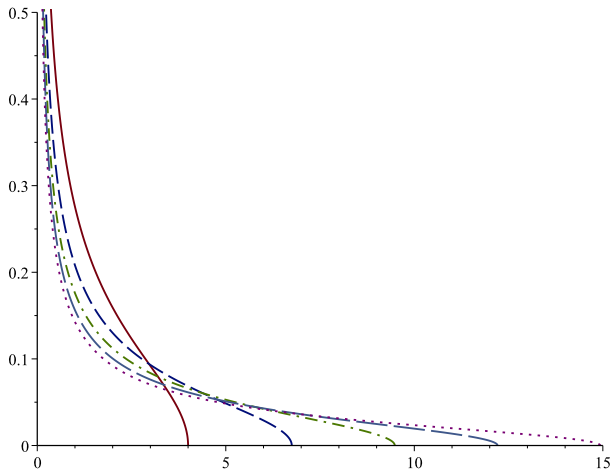
$$\lim_{n \rightarrow \infty} \frac{a_{\vec{n},j}}{n^{2\gamma}} = a_j, \quad \lim_{n \rightarrow \infty} \frac{b_{\vec{n},k}}{n^\gamma} = b_j.$$

Then uniformly on compact sets of $\mathbb{C} \setminus \mathbb{R}$

$$\lim_{n \rightarrow \infty} \frac{P_{\vec{n} + \vec{e}_k}(n^\gamma x)}{n^\gamma P_{\vec{n}}(n^\gamma x)} = z - b_k,$$

where z is the solution of the algebraic equation

$$x - z = \sum_{j=1}^r \frac{a_j}{z - b_j}, \quad \lim_{x \rightarrow \infty} z - x = 0.$$



$r = 1$ (solid), $r = 2$ (dash), $r = 3$ (dash dot), $r = 4$ (long dash), $r = 5$ (point)

Multiple Laguerre II

Type II multiple Laguerre polynomials of the second kind $L_{\vec{n}}^{\alpha, \vec{c}}(x)$

$$\int_0^{\infty} x^k L_{\vec{n}}^{\alpha, \vec{c}}(x) x^{\alpha} e^{-c_j x} dx = 0, \quad 0 \leq k \leq n_j - 1,$$

for $1 \leq j \leq r$.

Multiple Laguerre II

Type II multiple Laguerre polynomials of the second kind $L_{\vec{n}}^{\alpha, \vec{c}}(x)$

$$\int_0^{\infty} x^k L_{\vec{n}}^{\alpha, \vec{c}}(x) x^{\alpha} e^{-c_j x} dx = 0, \quad 0 \leq k \leq n_j - 1,$$

for $1 \leq j \leq r$.

Parameters $\alpha > -1$ and $c_j > 0$ with $c_i \neq c_j$ whenever $i \neq j$.

Multiple Laguerre II

Type II multiple Laguerre polynomials of the second kind $L_{\vec{n}}^{\alpha, \vec{c}}(x)$

$$\int_0^{\infty} x^k L_{\vec{n}}^{\alpha, \vec{c}}(x) x^{\alpha} e^{-c_j x} dx = 0, \quad 0 \leq k \leq n_j - 1,$$

for $1 \leq j \leq r$.

Parameters $\alpha > -1$ and $c_j > 0$ with $c_i \neq c_j$ whenever $i \neq j$.

Rodrigues formula:

$$(-1)^{|\vec{n}|} \prod_{j=1}^r c_j^{n_j} x^{\alpha} L_{\vec{n}}^{\alpha, \vec{c}}(x) = \prod_{j=1}^r \left(e^{c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{-c_j x} \right) x^{|\vec{n}| + \alpha}.$$

Explicit expression:

$$L_{\vec{n}}^{\alpha, \vec{c}}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} \binom{|\vec{n}| + \alpha}{|\vec{k}|} (-1)^{|\vec{k}|} \frac{|\vec{k}|!}{c_1^{k_1} \cdots c_r^{k_r}} x^{|\vec{n}| - |\vec{k}|}.$$

Explicit expression:

$$L_{\vec{n}}^{\alpha, \vec{c}}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} \binom{|\vec{n}| + \alpha}{|\vec{k}|} (-1)^{|\vec{k}|} \frac{|\vec{k}|!}{c_1^{k_1} \cdots c_r^{k_r}} x^{|\vec{n}| - |\vec{k}|}.$$

Recurrence relations:

$$xL_{\vec{n}}(x) = L_{\vec{n} + \vec{e}_k}(x) + b_{\vec{n}, k} L_{\vec{n}}(x) + \sum_{j=1}^r a_{\vec{n}, j} L_{\vec{n} - \vec{e}_j}(x),$$

$$a_{\vec{n}, j} = \frac{n_j (|\vec{n}| + \alpha)}{c_j^2}, \quad b_{\vec{n}, k} = \frac{|\vec{n}| + \alpha + 1}{c_k} + \sum_{j=1}^r \frac{n_j}{c_j}.$$

Multiple Laguerre II: differential properties

Raising operators:

$$\frac{d}{dx} x^{\alpha+1} e^{-c_j x} L_{\vec{n}-\vec{e}_j}^{\alpha+1, \vec{c}}(x) = -c_j x^{\alpha} e^{-c_j x} L_{\vec{n}}^{\alpha, \vec{c}}(x), \quad 1 \leq j \leq r.$$

Multiple Laguerre II: differential properties

Raising operators:

$$\frac{d}{dx} x^{\alpha+1} e^{-c_j x} L_{\vec{n}-\vec{e}_j}^{\alpha+1, \vec{c}}(x) = -c_j x^{\alpha} e^{-c_j x} L_{\vec{n}}^{\alpha, \vec{c}}(x), \quad 1 \leq j \leq r.$$

Lowering operator:

$$\frac{d}{dx} L_{\vec{n}}^{\alpha, \vec{c}}(x) = \sum_{j=1}^r n_j L_{\vec{n}-\vec{e}_j}^{\alpha+1, \vec{c}}(x).$$

Multiple Laguerre II: differential properties

Raising operators:

$$\frac{d}{dx} x^{\alpha+1} e^{-c_j x} L_{\vec{n}-\vec{e}_j}^{\alpha+1, \vec{c}}(x) = -c_j x^{\alpha} e^{-c_j x} L_{\vec{n}}^{\alpha, \vec{c}}(x), \quad 1 \leq j \leq r.$$

Lowering operator:

$$\frac{d}{dx} L_{\vec{n}}^{\alpha, \vec{c}}(x) = \sum_{j=1}^r n_j L_{\vec{n}-\vec{e}_j}^{\alpha+1, \vec{c}}(x).$$

Differential equation:

$$\left(\prod_{j=1}^r D_j \right) x^{\alpha+1} D L_{\vec{n}}^{\alpha, \vec{c}}(x) = - \sum_{j=1}^r c_j n_j \left(\prod_{i \neq j} D_i \right) x^{\alpha} L_{\vec{n}}^{\alpha, \vec{c}}(x),$$

$$D = \frac{d}{dx}, \quad D_j = e^{c_j x} D e^{-c_j x}.$$

Multiple Laguerre II: integral representations

Integral representation for type II: (integer α)

$$x^\alpha L_{\vec{n}}^{\alpha, \vec{c}}(x) = (-1)^{|\vec{n}|} \frac{(|\vec{n}| + \alpha)!}{c_1^{n_1} \cdots c_r^{n_r}} \frac{1}{2\pi i} \oint_{\Gamma} e^{xs} s^{-|\vec{n}| - \alpha - 1} \prod_{j=1}^r (s - c_j)^{n_j} ds,$$

Γ a contour around 0 which does not encircle any of the c_j .

Multiple Laguerre II: integral representations

Integral representation for type II: (integer α)

$$x^\alpha L_{\vec{n}}^{\alpha, \vec{c}}(x) = (-1)^{|\vec{n}|} \frac{(|\vec{n}| + \alpha)!}{c_1^{n_1} \cdots c_r^{n_r}} \frac{1}{2\pi i} \oint_{\Gamma} e^{xs} s^{-|\vec{n}| - \alpha - 1} \prod_{j=1}^r (s - c_j)^{n_j} ds,$$

Γ a contour around 0 which does not encircle any of the c_j .

Integral representation for type I: (α integer)

$$e^{-c_j x} A_{\vec{n}, k}(x) = (-1)^{|\vec{n}| + 1} \frac{c_1^{n_1} \cdots c_r^{n_r}}{(|\vec{n}| + \alpha - 1)!} \frac{1}{2\pi i} \oint_{\Gamma_k} e^{-xt} t^{|\vec{n}| + \alpha - 1} \prod_{j=1}^r (t - c_j)^{-n_j}$$

Γ_k a contour around c_k , does not enclose 0 or any other c_j ($j \neq k$).

Multiple Laguerre II: integral representations

Integral representation for type II: (integer α)

$$x^\alpha L_{\vec{n}}^{\alpha, \vec{c}}(x) = (-1)^{|\vec{n}|} \frac{(|\vec{n}| + \alpha)!}{c_1^{n_1} \cdots c_r^{n_r}} \frac{1}{2\pi i} \oint_{\Gamma} e^{xs} s^{-|\vec{n}| - \alpha - 1} \prod_{j=1}^r (s - c_j)^{n_j} ds,$$

Γ a contour around 0 which does not encircle any of the c_j .

Integral representation for type I: (α integer)

$$e^{-c_j x} A_{\vec{n}, k}(x) = (-1)^{|\vec{n}|+1} \frac{c_1^{n_1} \cdots c_r^{n_r}}{(|\vec{n}| + \alpha - 1)!} \frac{1}{2\pi i} \oint_{\Gamma_k} e^{-xt} t^{|\vec{n}| + \alpha - 1} \prod_{j=1}^r (t - c_j)^{-n_j} dt$$

Γ_k a contour around c_k , does not enclose 0 or any other c_j ($j \neq k$).

$$Q_{\vec{n}}(x) = (-1)^{|\vec{n}|+1} \frac{c_1^{n_1} \cdots c_r^{n_r}}{(|\vec{n}| + \alpha - 1)!} \frac{1}{2\pi i} \oint_{\Gamma} e^{-xt} t^{|\vec{n}| + \alpha - 1} \prod_{j=1}^r (t - c_j)^{-n_j} dt,$$

Γ a contour encircling c_1, \dots, c_r but not 0.



Random matrices: Wishart ensemble

John Wishart (1928) introduced the **Wishart distribution** for $N \times N$ positive definite Hermitian matrices

$$M = XX^*, \quad X \in \mathbb{C}^{N \times (N+p)}$$

where all the columns of X are independent and have a multivariate Gauss distribution with covariance matrix Σ .

Random matrices: Wishart ensemble

John Wishart (1928) introduced the **Wishart distribution** for $N \times N$ positive definite Hermitian matrices

$$M = XX^*, \quad X \in \mathbb{C}^{N \times (N+p)}$$

where all the columns of X are independent and have a multivariate Gauss distribution with covariance matrix Σ .

$$\frac{1}{Z_N} e^{-\text{Tr}(\Sigma^{-1}M)} (\det M)^p dM.$$

Random matrices: Wishart ensemble

John Wishart (1928) introduced the **Wishart distribution** for $N \times N$ positive definite Hermitian matrices

$$M = XX^*, \quad X \in \mathbb{C}^{N \times (N+p)}$$

where all the columns of X are independent and have a multivariate Gauss distribution with covariance matrix Σ .

$$\frac{1}{Z_N} e^{-\text{Tr}(\Sigma^{-1}M)} (\det M)^p dM.$$

If $\Sigma = I_N$ then Laguerre polynomials (with $\alpha = p$) play an important role.

Random matrices: Wishart ensemble

John Wishart (1928) introduced the **Wishart distribution** for $N \times N$ positive definite Hermitian matrices

$$M = XX^*, \quad X \in \mathbb{C}^{N \times (N+p)}$$

where all the columns of X are independent and have a multivariate Gauss distribution with covariance matrix Σ .

$$\frac{1}{Z_N} e^{-\text{Tr}(\Sigma^{-1}M)} (\det M)^p dM.$$

If $\Sigma = I_N$ then Laguerre polynomials (with $\alpha = p$) play an important role.

If Σ^{-1} has eigenvalues c_1, \dots, c_r with multiplicities n_1, \dots, n_r , then we need multiple Laguerre polynomials of the second kind:

Random matrices: Wishart ensemble

John Wishart (1928) introduced the **Wishart distribution** for $N \times N$ positive definite Hermitian matrices

$$M = XX^*, \quad X \in \mathbb{C}^{N \times (N+p)}$$

where all the columns of X are independent and have a multivariate Gauss distribution with covariance matrix Σ .

$$\frac{1}{Z_N} e^{-\text{Tr}(\Sigma^{-1}M)} (\det M)^p dM.$$

If $\Sigma = I_N$ then Laguerre polynomials (with $\alpha = p$) play an important role.

If Σ^{-1} has eigenvalues c_1, \dots, c_r with multiplicities n_1, \dots, n_r , then we need multiple Laguerre polynomials of the second kind:

$$\mathbb{E} \left(\det(M - zI_N) \right) = (-1)^{|\vec{n}|} L_{\vec{n}}^{p, \vec{c}}(z).$$

Multiple Laguerre II: asymptotics

Riemann-Hilbert approach (Lysov and Wielonsky, 2008) for $r = 2$.

Asymptotic behavior for $P_n(z) = n^{-2n} L_{n,n}^{\alpha, (c_1, c_2)}(nz)$

Multiple Laguerre II: asymptotics

Riemann-Hilbert approach (Lysov and Wielonsky, 2008) for $r = 2$.

Asymptotic behavior for $P_n(z) = n^{-2n} L_{n,n}^{\alpha, (c_1, c_2)}(nz)$

Cubic equation

$$z = \frac{2c_1 c_2 - (c_1 + c_2)\psi}{\psi(\psi - c_1)(\psi - c_2)}$$

Multiple Laguerre II: asymptotics

Riemann-Hilbert approach (Lysov and Wielonsky, 2008) for $r = 2$.

Asymptotic behavior for $P_n(z) = n^{-2n} L_{n,n}^{\alpha, (c_1, c_2)}(nz)$

Cubic equation

$$z = \frac{2c_1 c_2 - (c_1 + c_2)\psi}{\psi(\psi - c_1)(\psi - c_2)}$$

Three solutions, which for $z \rightarrow \infty$

$$\psi_0(z) = \frac{2}{z} + \mathcal{O}(1/z^2),$$

$$\psi_1(z) = c_1 - \frac{1}{z} + \mathcal{O}(1/z^2),$$

$$\psi_2(z) = c_2 - \frac{1}{z} + \mathcal{O}(1/z^2).$$

Multiple Laguerre II: asymptotics

Riemann-Hilbert approach (Lysov and Wielonsky, 2008) for $r = 2$.

Asymptotic behavior for $P_n(z) = n^{-2n} L_{n,n}^{\alpha, (c_1, c_2)}(nz)$

Cubic equation

$$z = \frac{2c_1 c_2 - (c_1 + c_2)\psi}{\psi(\psi - c_1)(\psi - c_2)}$$

Three solutions, which for $z \rightarrow \infty$

$$\psi_0(z) = \frac{2}{z} + \mathcal{O}(1/z^2),$$

$$\psi_1(z) = c_1 - \frac{1}{z} + \mathcal{O}(1/z^2),$$

$$\psi_2(z) = c_2 - \frac{1}{z} + \mathcal{O}(1/z^2).$$

Zero distribution ν_1 determined by ψ_0 :

$$\int \log(z - x) d\nu_1(x) = \int^z \psi_0(s) ds$$

Multiple Laguerre II: asymptotics

Let

$$\kappa = \frac{7 + 3\sqrt{3}}{2} + \sqrt{\frac{36 + 21\sqrt{3}}{2}} = 12.1136\dots$$

Multiple Laguerre II: asymptotics

Let

$$\kappa = \frac{7 + 3\sqrt{3}}{2} + \sqrt{\frac{36 + 21\sqrt{3}}{2}} = 12.1136\dots$$

- $1 < c_2/c_1 < \kappa$: zeros accumulate on one interval $[0, d]$.

Multiple Laguerre II: asymptotics

Let

$$\kappa = \frac{7 + 3\sqrt{3}}{2} + \sqrt{\frac{36 + 21\sqrt{3}}{2}} = 12.1136\dots$$

- $1 < c_2/c_1 < \kappa$: zeros accumulate on one interval $[0, d]$.
- $c_2/c_1 > \kappa$: zeros accumulate on two intervals $[0, b] \cup [c, d]$.

Multiple Laguerre II: asymptotics

Let

$$\kappa = \frac{7 + 3\sqrt{3}}{2} + \sqrt{\frac{36 + 21\sqrt{3}}{2}} = 12.1136\dots$$

- $1 < c_2/c_1 < \kappa$: zeros accumulate on one interval $[0, d]$.
- $c_2/c_1 > \kappa$: zeros accumulate on two intervals $[0, b] \cup [c, d]$.
- $c_2/c_1 = \kappa$: $b = c$ (phase transition: birth of a gap).

Multiple Laguerre II: asymptotics

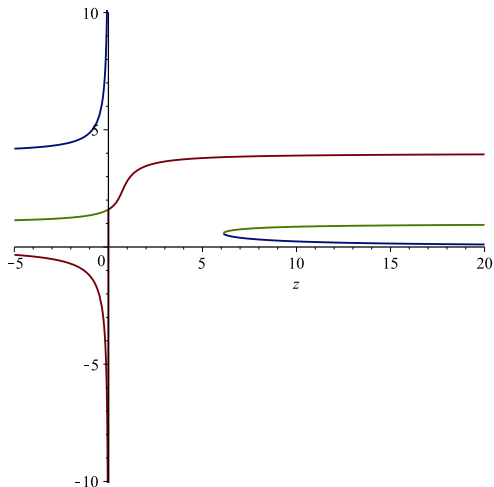
Let

$$\kappa = \frac{7 + 3\sqrt{3}}{2} + \sqrt{\frac{36 + 21\sqrt{3}}{2}} = 12.1136\dots$$

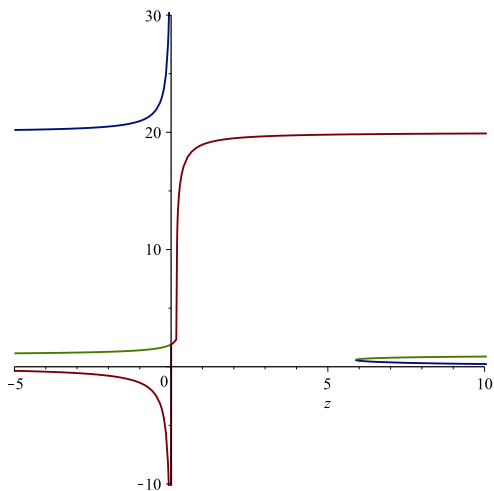
- $1 < c_2/c_1 < \kappa$: zeros accumulate on one interval $[0, d]$.
- $c_2/c_1 > \kappa$: zeros accumulate on two intervals $[0, b] \cup [c, d]$.
- $c_2/c_1 = \kappa$: $b = c$ (phase transition: birth of a gap).

$0, b, c, d$ are the (real) roots of the discriminant of the cubic equation

$$c_1^2 c_2^2 (c_1 - c_2)^2 z^4 - 6c_1 c_2 (c_1 + c_2) (c_1 - c_2)^2 z^3 + (c_1^4 + 28c_1^3 c_2 - 54c_1^2 c_2^2 + 28c_1 c_2^3 + c_2^4) z^2 - 4(c_1 + c_2)^3 z = 0.$$



The functions ψ_0, ψ_1, ψ_2 for $c_1 = 1, c_2 = 4$ (interval is $[0, 6.12]$)



The functions ψ_0, ψ_1, ψ_2 for $c_1 = 1, c_2 = 20$ (intervals are $[0, 0.18] \cup [0.23, 5.87]$)

Multiple Laguerre II: asymptotics (two intervals)

Theorem (Lysov-Wielonsky)

Uniformly for z on compact subsets of $\mathbb{C} \setminus ([0, b] \cup [c, d])$

$$P_n(z) = -2^{\alpha+1/2} i \frac{\psi_0(z) - c_1}{(z\psi_0(z))^\alpha \sqrt{D(\psi_0(z))}} (\psi_0(z) - c_2) e^{n(\lambda_0(z) - \ell_0)} \left(1 + \mathcal{O}(1/n)\right),$$

ℓ_0 is a constant such that

$$\lambda_0(z) = 2 \log z + \ell_0 + \mathcal{O}(1/z),$$

and D is the polynomial

$$D(x) = 2(c_1 + c_2)x^3 - (c_1^2 + 8c_1c_2 + c_2^2)x^2 + 4(c_1 + c_2)c_1c_2x - 2c_1^2c_2^2.$$

Multiple Laguerre II: asymptotics

Theorem (Lysov-Wielonsky)

On the intervals $[\epsilon, b - \epsilon] \cup [c + \epsilon, d - \epsilon]$ (with $\epsilon > 0$)

$$P_n(x) = 2^{\alpha+1/2} x^{-\alpha} \left(A(x) \sin[n\Im\lambda_0^+(x) + \varphi(x)] + \mathcal{O}(1/n) \right) e^{n(\Re\lambda_0^+(x) - \ell_0)},$$

where

$$A(x) = 2 \left| \frac{(\psi_0^+(x) - c_1)(\psi_0^+(x) - c_2)}{(\psi_0^+)^{\alpha}(x) \sqrt{D(\psi_0^+(x))}} \right|,$$

and

$$\varphi(x) = \arg \frac{(\psi_0^+(x) - c_1)(\psi_0^+(x) - c_2)}{(\psi_0^+)^{\alpha}(x) \sqrt{D(\psi_0^+(x))}}.$$

Theorem (Lysov-Wielonsky)

If z is in a neighborhood of d , then

$$P_n(z) = -2^{\alpha+1/2} \sqrt{\pi} \left(n^{1/6} B_d(z) \text{Ai}(n^{2/3} f_d(z)) (1 + \mathcal{O}(1/n)) \right. \\ \left. + n^{-1/6} C_d(z) \text{Ai}'(n^{2/3} f_d(z)) (1 + \mathcal{O}(1/n)) \right) e^{n(\lambda_0(z) + \lambda_1(z) - 2\ell_0)/2},$$

where Ai is the Airy function, B_d and C_d are analytic functions in a neighborhood of d , and

$$\frac{4}{3} [f_d(z)]^{3/2} = \lambda_1(z) - \lambda_0(z).$$

Theorem (Lysov-Wielonsky)

If z is in a neighborhood of d , then

$$P_n(z) = -2^{\alpha+1/2} \sqrt{\pi} \left(n^{1/6} B_d(z) \text{Ai}(n^{2/3} f_d(z)) (1 + \mathcal{O}(1/n)) \right. \\ \left. + n^{-1/6} C_d(z) \text{Ai}'(n^{2/3} f_d(z)) (1 + \mathcal{O}(1/n)) \right) e^{n(\lambda_0(z) + \lambda_1(z) - 2\ell_0)/2},$$

where Ai is the Airy function, B_d and C_d are analytic functions in a neighborhood of d , and

$$\frac{4}{3} [f_d(z)]^{3/2} = \lambda_1(z) - \lambda_0(z).$$

Similar results for b and c .





Theorem (Lysov-Wielonsky)

If z is in a neighborhood of 0, then

$$P_n(z) = -2^{\alpha+1/2} i \sqrt{\pi n f_0(z)}^{1/4} e^{-\alpha\pi/2} \left(B_0(z) J_\alpha(2n(-f_0(z))^{1/2}) (1 + \mathcal{O}(1/n)) \right. \\ \left. + C_0(z) J'_\alpha(2n(-f_0(z))^{1/2}) (1 + \mathcal{O}(1/n)) \right) e^{n(\lambda_0(z) + \lambda_2(z) - 2\ell_0)/2},$$

where J_α is the Bessel function of order α , B_0 and C_0 are analytic functions in a neighborhood of 0, and

$$-4[f_0(z)]^{1/2} = \lambda_2(z) - \lambda_0(z) + 4\pi i.$$

-  P.M. Bleher, A.B.J. Kuijlaars, **Integral representations for multiple Hermite and multiple Laguerre polynomials**, Ann. Inst. Fourier, Grenoble **55** (2005), no. 6, 2001–2014.
-  M.E.H. Ismail, **Classical and Quantum Orthogonal Polynomials in One Variable**, Encyclopedia of Mathematics and its Applications 98, Cambridge University Press, 2005 (paperback edition 2009) ; in particular **Chapter 23**.
-  V. Lysov, F. Wielonsky, **Strong asymptotic for multiple Laguerre polynomials**, Constr. Approx. **28** (2008), no. 1, 61–111.
-  T. Neuschel, W. Van Assche, **Asymptotic zero distribution of Jacobi-Piñeiro and multiple Laguerre polynomials**, J. Approx. Theory **205** (2016), 114–132.