

Multiple Orthogonal Polynomials

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Summer school on OPSF, University of Kent

26–30 June, 2017

Plan of the course

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lecture 1: Definitions + basic properties

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Jacobi-Angelesco + Jacobi-Piñeiro

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- lecture 2: Hermite-Padé, Multiple Hermite polynomials**
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- lecture 5: Riemann-Hilbert problem

Hermite-Padé approximation

Let (f_1, \dots, f_r) be r Markov functions, i.e.,

$$f_j(z) = \int \frac{d\mu_j(x)}{z-x} = \sum_{k=0}^{\infty} \frac{m_k^{(j)}}{z^{k+1}}.$$

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Definition (Type I Hermite-Padé)

Type I Hermite-Padé approximation is to find r polynomials $(A_{\vec{n},1}, \dots, A_{\vec{n},r})$, with $\deg A_{\vec{n},j} \leq n_j - 1$, and a polynomial $B_{\vec{n}}$ such that

$$\sum_{j=1}^r A_{\vec{n},j}(z) f_j(z) - B_{\vec{n}}(z) = \mathcal{O}\left(\frac{1}{z^{|\vec{n}|}}\right), \quad z \rightarrow \infty.$$

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$$B_{\vec{n}}(z) = \int \sum_{j=1}^r \frac{A_{\vec{n},j}(z) - A_{\vec{n},j}(x)}{z-x} d\mu_j(x).$$

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Definition (Type II Hermite-Padé)

Type II Hermite-Padé approximation is to find a polynomial $P_{\vec{n}}$ of degree $\leq |\vec{n}|$ and polynomials $Q_{\vec{n},1}, \dots, Q_{\vec{n},r}$ such that

$$P_{\vec{n}}(z)f_j(z) - Q_{\vec{n},j}(z) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right), \quad z \rightarrow \infty,$$

for $1 \leq j \leq r$.

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Multiple Hermite polynomials

The type II multiple Hermite polynomials $H_{\vec{n}}$ satisfy

$$\int_{-\infty}^{\infty} H_{\vec{n}}(x) x^k e^{-x^2 + c_j x} dx = 0, \quad 0 \leq k \leq n_j - 1$$

for $1 \leq j \leq r$, with $c_i \neq c_j$ whenever $i \neq j$.

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Rodrigues formula:

$$e^{-x^2} H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \left(\prod_{j=1}^r e^{-c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{c_j x} \right) e^{-x^2}.$$

Multiple Hermite polynomials

Explicit expression:

$$H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} c_1^{n_1-k_1} \cdots c_r^{n_r-k_r} (-1)^{|\vec{k}|} H_{|\vec{k}|}(x),$$

where H_n are the usual Hermite polynomials.

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Nearest neighbor recurrence relations:

$$xH_{\vec{n}}(x) = H_{\vec{n}+\vec{e}_k}(x) + \frac{c_k}{2} H_{\vec{n}}(x) + \frac{1}{2} \sum_{j=1}^r n_j H_{\vec{n}-\vec{e}_j}(x), \quad 1 \leq k \leq r.$$

Raising operators:

$$\left(e^{-x^2+c_jx} H_{\vec{n}-\vec{e}_j}(x) \right)' = -2e^{-x^2+c_jx} H_{\vec{n}}(x), \quad 1 \leq j \leq r.$$

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Lowering operator:

$$H'_{\vec{n}}(x) = \sum_{j=1}^r n_j H_{\vec{n}-\vec{e}_j}(x).$$

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Differential equation:

$$\left(\prod_{j=1}^r D_j \right) D H_{\vec{n}}(x) = -2 \left(\sum_{j=1}^r n_j \prod_{i \neq j} D_i \right) H_{\vec{n}}(x),$$

where

$$D = \frac{d}{dx}, \quad D_j = e^{x^2-c_jx} D e^{-x^2+c_jx}$$

Type II multiple Hermite:

$$H_{\vec{n}}(x) = \frac{1}{\sqrt{\pi}i} \int_{-i\infty}^{i\infty} e^{(s-x)^2} \prod_{j=1}^r \left(s - \frac{c_j}{2}\right)^{n_j} ds.$$

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Type I multiple Hermite:

$$e^{-x^2+c_k x} A_{\vec{n},k}(x) = \frac{1}{\sqrt{\pi}2\pi i} \oint_{\Gamma_k} e^{-(t-x)^2} \prod_{j=1}^r \left(t - \frac{c_j}{2}\right)^{-n_j} dt$$

where Γ_k is a closed contour encircling $c_k/2$ once and none of the other $c_j/2$.

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where Γ_k is a closed contour encircling $c_k/2$ once and none of the other $c_j/2$.

$$Q_{\vec{n}}(x) = \sum_{k=1}^r e^{-x^2+c_k x} A_{\vec{n},k}(x) = \frac{1}{\sqrt{\pi}2\pi i} \oint_{\Gamma} e^{-(t-x)^2} \prod_{j=1}^r \left(t - \frac{c_j}{2}\right)^{-n_j} dt$$

where Γ is a closed contour encircling all $c_j/2$.

Random matrices with external source

Let M be a random Hermitian matrix of size $N \times N$, and consider the **ensemble** with probability distribution

$$\frac{1}{Z_N} \exp\left(-\text{Tr}(M^2 - AM)\right) dM, \quad dM = \prod_{i=1}^N dM_{i,i} \prod_{1 \leq i < j \leq N} dM_{i,j}$$

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Property

Suppose A has eigenvalues c_1, \dots, c_r with multiplicities n_1, \dots, n_r , then

$$\mathbb{E}\left(\det(M - zI_N)\right) = (-1)^{|\vec{n}|} H_{\vec{n}}(z).$$

Property

The density of the eigenvalues is given by

$$P_N(\lambda_1, \dots, \lambda_N) = \frac{1}{N!} \det \left(K_N(\lambda_i, \lambda_j) \right)_{i,j=1}^N,$$

where the kernel is given by

$$K_N(x, y) = e^{-(x^2+y^2)/2} \sum_{k=0}^{N-1} H_{\vec{n}_k}(x) Q_{\vec{n}_{k+1}}(y),$$

with $(\vec{n}_k)_{0 \leq k \leq N}$ a path from $\vec{0}$ to \vec{n} in \mathbb{N}^r and

$$Q_{\vec{n}}(y) = \sum_{j=1}^r A_{\vec{n},j}(y) e^{c_j y}.$$

Property

The m -point correlation function

$$R_m(\lambda_1, \dots, \lambda_m) = \frac{N!}{(N-m)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_N(\lambda_1, \dots, \lambda_N) d\lambda_{m+1} \cdots d\lambda_N$$

is given by

$$R_m(\lambda_1, \dots, \lambda_m) = \det \left(K_N(\lambda_i, \lambda_j) \right)_{i,j=1}^m,$$

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Non-intersecting Brownian motions

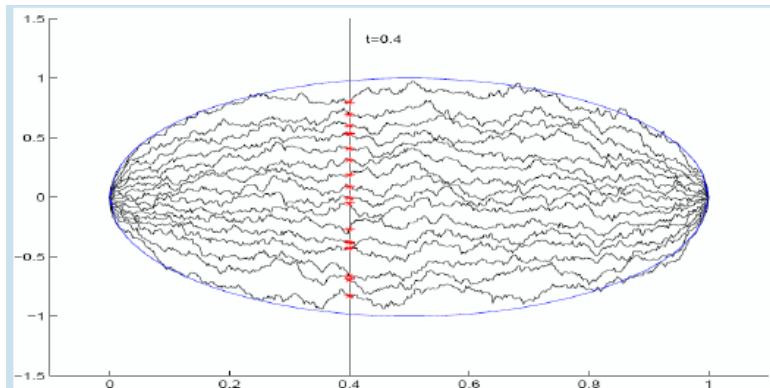


Figure: Non-intersecting Brownian motions

Non-intersecting Brownian motions

Density of the probability that the n non-intersecting paths, leaving ($t = 0$) at a_1, \dots, a_n and arriving ($t = 1$) at b_1, \dots, b_n are at x_1, \dots, x_n at time $t \in (0, 1)$ is¹

$$\rho_{n,t}(x_1, \dots, x_n) = \frac{1}{Z_n} \det \left(P(t, a_j, x_k) \right)_{j,k=1}^n \det \left(P(1-t, b_j, x_k) \right)_{j,k=1}^n$$
$$P(t, a, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x-a)^2}.$$

¹S. Karlin, J. McGregor: *Coincidence probabilities*, Pacific J. Math. **9** (1959), 1141–1164

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$$P(t, a, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x-a)^2}.$$

When $a_1, \dots, a_n \rightarrow 0$ and $b_1, \dots, b_n \rightarrow 0$ then

$$p_{n,t}(x_1, \dots, x_n) = \frac{1}{n!} \det \left(K_n(x_j, x_k) \right)_{j,k=1}^n$$
$$K_n(x, y) = e^{-\frac{x^2}{4t} - \frac{y^2}{4(1-t)}} \sum_{k=0}^{n-1} H_k \left(\frac{x}{\sqrt{2t}} \right) H_k \left(\frac{y}{\sqrt{2(1-t)}} \right)$$

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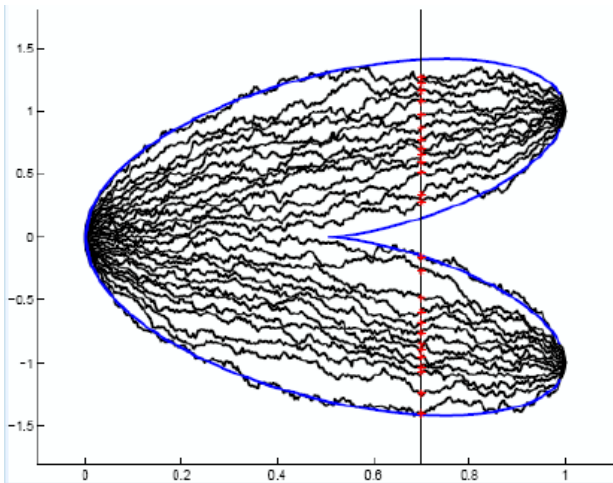


Figure: Non-intersecting Brownian motions (two arriving points)

Non-intersecting Brownian motions

When $a_1, \dots, a_n \rightarrow 0$ and $b_1, \dots, b_{n/2} \rightarrow -b$, $b_{n/2+1}, \dots, b_n \rightarrow b$
then

$$p_{n,t}(x_1, \dots, x_n) = \frac{1}{n!} \det \left(K_n(x_j, x_k) \right)_{j,k=1}^n$$

$$K_n(x, y) = e^{-\frac{x^2}{4t} - \frac{y^2}{4(1-t)}} \sum_{k=0}^{n-1} H_{\vec{n}_k} \left(\frac{x}{\sqrt{2t}} \right) Q_{\vec{n}_{k+1}} \left(\frac{y}{\sqrt{2(1-t)}} \right)$$

with multiple orthogonal polynomials for the weights

$$e^{-x^2-2bx}, \quad e^{-x^2+2bx}$$

Non-intersecting squared Bessel paths

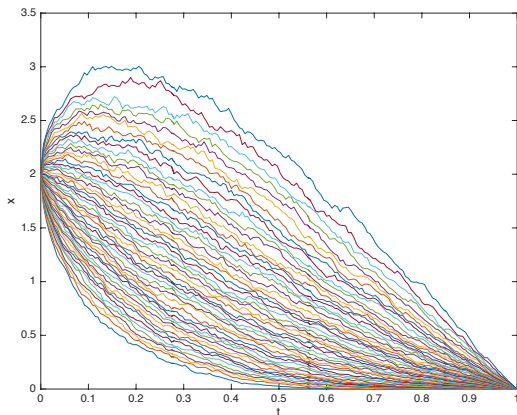







Figure: Non-intersecting squared Bessel paths

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