Multiple Orthogonal Polynomials

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Introduction

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- Location of zeros
- Three term recurrence relation:

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- Classical orthogonal polynomials: Jacobi - Laguerre - Hermite

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lecture 5: Riemann-Hilbert problem

Definition: type I MOPS

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We use **multi-indices** $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ and denote their **length** by $|\vec{n}| = n_1 + n_2 + \cdots + n_r$.

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Definition (type I)

Type I multiple orthogonal polynomials for $\vec{n}$ consist of the vector $(A_{\vec{n},1}, \ldots, A_{\vec{n},r})$ of r polynomials, with $\deg A_{\vec{n},j} \leq n_j - 1$, for which

$$\int x^k \sum_{j=1}^{r} A_{\vec{n},j}(x)\, d\mu_j(x) = 0, \qquad 0 \leq k \leq |\vec{n}| - 2,$$

with normalization

$$\int x^{|\vec{n}|-1} \sum_{j=1}^{r} A_{\vec{n},j}(x)\, d\mu_j(x) = 1.$$

Definition: type II MOPS

Definition (type II)

The type II multiple orthogonal polynomial for $\vec{n}$ is the **monic** polynomial $P_{\vec{n}}$ of degree $|\vec{n}|$ for which

$$\int x^k P_{\vec{n}}(x)\, d\mu_j(x) = 0, \qquad 0 \leq k \leq n_j - 1,$$

for $1 \leq j \leq r$.

Normal indices

a multi-index $\vec{n}$ is **normal** if the type I vector $(A_{\vec{n},1}, \ldots A_{\vec{n},r})$ exists and is unique

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$$\det \begin{pmatrix} M_{n_1}^{(1)} \\ M_{n_2}^{(2)} \\ \vdots \\ M_{n_r}^{(r)} \end{pmatrix} \neq 0, \qquad M_{n_j}^{(j)} = \begin{pmatrix} m_0^{(j)} & m_1^{(j)} & \cdots & m_{|\vec{n}|-1}^{(j)} \\ m_1^{(j)} & m_2^{(j)} & \cdots & m_{|\vec{n}|}^{(j)} \\ \vdots & \vdots & \cdots & \vdots \\ m_{n_j-1}^{(j)} & m_{n_j}^{(j)} & \cdots & m_{|\vec{n}|+n_j-2}^{(j)} \end{pmatrix},$$

$$m_k^{(j)} = \int x^k \, d\mu_j(x).$$

Special systems: Angelesco systems

Definition (Angelesco system)

The measures $(\mu_1, \ldots, \mu_r)$ are an **Angelesco system** if the supports of the measures are subsets of disjoint intervals Δ_j, i.e., $\text{supp}(\mu_j) \subset \Delta_j$ and $\Delta_i \cap \Delta_j = \emptyset$ whenever $i \neq j$.

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Usually one allows that the intervals are touching, i.e., $\mathring{\Delta}_i \cap \mathring{\Delta}_j = \emptyset$ whenever $i \neq j$.

Special systems: Angelesco systems

Theorem (Angelesco, Nikishin)

The type II multiple orthogonal polynomial $P_{\vec{n}}$ has exactly n_j distinct zeros on $\mathring{\Delta}_j$ for $1 \leq j \leq r$.

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Every multi-index $\vec{n}$ is normal (an Angelesco system is perfect).

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Exercise

Show that every $A_{\vec{n},j}$ has $n_j - 1$ zeros on $\mathring{\Delta}_j$.

Special systems: AT systems

Definition

The functions $\varphi_1, \ldots, \varphi_n$ are a **Chebyshev system** on $[a, b]$ if every linear combination $\sum_{i=1}^n a_i \varphi_i$ with $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$ has at most $n - 1$ zeros on $[a, b]$.

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Definition (AT-system)

The measures $(\mu_1, \ldots, \mu_r)$ are an **AT-system** on the interval $[a, b]$ if the measures are all absolutely continuous with respect to a positive measure μ on $[a, b]$, i.e., $d\mu_j(x) = w_j(x)\, d\mu(x)$ $(1 \leq j \leq r)$, and for every $\vec{n}$ the functions

$$w_1(x), xw_1(x), \ldots, x^{n_1-1}w_1(x),\ w_2(x), xw_2(x), \ldots, x^{n_2-1}w_2(x), \\ \ldots, w_r(x), xw_r(x), \ldots, x^{n_r-1}w_r(x)$$

are a Chebyshev system on $[a, b]$.

Special systems: AT-systems

Theorem

For an AT-system the function

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x) w_j(x)$$

has exactly $|\vec{n}| - 1$ *sign changes on* (a, b).

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If $(\mu_1, \ldots, \mu_r)$ is an AT-system, then the type II multiple orthogonal polynomial $P_{\vec{n}}$ has exactly $|\vec{n}|$ distinct zeros on (a, b).

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Theorem

If $(\mu_1, \ldots, \mu_r)$ *is an AT-system, then the type II multiple orthogonal polynomial* $P_{\vec{n}}$ *has exactly* $|\vec{n}|$ *distinct zeros on* (a, b).

Corollary

Every multi-index in an AT-system is normal (an AT-system is perfect).

Special systems: Nikishin systems

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Definition (Nikishin system for $r = 2$)

A Nikishin system of order $r = 2$ consists of two measures (μ_1, μ_2), both supported on an interval Δ_2, and such that

$$\frac{d\mu_2(x)}{d\mu_1(x)} = \int_{\Delta_1} \frac{d\sigma(t)}{x - t},$$

where σ is a positive measure on an interval Δ_1 and $\Delta_1 \cap \Delta_2 = \emptyset$.

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Theorem (Nikishin, Driver-Stahl)

A Nikishin system of order two is perfect.

Notation: $\langle\sigma_1, \sigma_2\rangle$ is a measure which is absolutely continuous with respect to σ_1 and for which the Radon-Nikodym derivative is a Stieltjes transform of σ_2:

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Definition (Nikishin system for general r)

A Nikishin system of order r on an interval Δ_r is a system of r measures $(\mu_1, \mu_2, \ldots, \mu_r)$ supported on Δ_r such that $\mu_j = \langle \mu_1, \sigma_j \rangle$ $(2 \leq j \leq r)$, where $(\sigma_2, \ldots, \sigma_r)$ is a Nikishin system of order $r-1$ on an interval Δ_{r-1} and $\Delta_r \cap \Delta_{r-1} = \emptyset$.

Special systems: Nikishin systems

Theorem (Fidalgo Prieto and López Lagomasino)

Every Nikishin system is perfect.

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Every Nikishin system is perfect.

Proof.

Ask Guillermo

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Every Nikishin system is perfect.

Proof.

Ask Guillermo or Ulises. □

Biorthogonality

In most cases the measures $(\mu_1, \ldots, \mu_r)$ are absolutely continuous with respect to one fixed measure μ:

$$d\mu_j(x) = w_j(x)\, d\mu(x), \qquad 1 \leq j \leq r.$$

We then define the **type I function**

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x) w_j(x).$$

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Property (biorthogonality)

$$\int P_{\vec{n}}(x) Q_{\vec{m}}(x)\, d\mu(x) = \begin{cases} 0, & \text{if } \vec{m} \leq \vec{n}, \\ 0, & \text{if } |\vec{n}| \leq |\vec{m}| - 2, \\ 1, & \text{if } |\vec{n}| = |\vec{m}| - 1. \end{cases}$$

Recurrence relations

Nearest neighbor recurrence relations for type II MOPS

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_1}(x) + b_{\vec{n},1}P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x),$$

$$\vdots$$

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_r}(x) + b_{\vec{n},r}P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x).$$

$$\vec{e}_j = (0,\ldots,0,\overbrace{1}^{j},0,\ldots,0)$$

Recurrence relations

Nearest neighbor recurrence relations for type I MOPS

$$xQ_{\vec{n}}(x) = Q_{\vec{n}-\vec{e}_1}(x) + b_{\vec{n}-\vec{e}_1,1} Q_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j} Q_{\vec{n}+\vec{e}_j}(x),$$

$$\vdots$$

$$xQ_{\vec{n}}(x) = Q_{\vec{n}-\vec{e}_r}(x) + b_{\vec{n}-\vec{e}_r,r} Q_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j} Q_{\vec{n}+\vec{e}_j}(x).$$

Recurrence relations

Theorem (Van Assche)

The recurrence coefficients $(a_{\vec{n},1},\ldots,a_{\vec{n},r})$ *and* $(b_{\vec{n},1},\ldots,b_{\vec{n},r})$ *satisfy the partial difference equations*

$$\begin{aligned}
b_{\vec{n}+\vec{e}_i,j} - b_{\vec{n},j} &= b_{\vec{n}+\vec{e}_j,i} - b_{\vec{n},i} \\
\sum_{k=1}^{r} a_{\vec{n}+\vec{e}_j,k} - \sum_{k=1}^{r} a_{\vec{n}+\vec{e}_i,k} &= \det\begin{pmatrix} b_{\vec{n}+\vec{e}_j,i} & b_{\vec{n},i} \\ b_{\vec{n}+\vec{e}_i,j} & b_{\vec{n},j} \end{pmatrix}, \\
\frac{a_{\vec{n},i}}{a_{\vec{n}+\vec{e}_j,i}} &= \frac{b_{\vec{n}-\vec{e}_i,j} - b_{\vec{n}-\vec{e}_i,i}}{b_{\vec{n},j} - b_{\vec{n},i}}
\end{aligned}$$

for all $1 \leq i \neq j \leq r$.

Recurrence relations

Let $(\vec{n}_k)_{k\geq 0}$ be a path in $\mathbb{N}^r$ starting from $\vec{n}_0 = \vec{0}$, such that $\vec{n}_{k+1} - \vec{n}_k = \vec{e}_i$ for some $1 \leq i \leq r$. Then

$$xP_{\vec{n}_k}(x) = P_{\vec{n}_{k+1}}(x) + \sum_{j=0}^{r} \beta_{\vec{n}_k,j} P_{\vec{n}_{k-j}}(x).$$

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An important case is the **stepline**:

$$\vec{n}_k = (\overbrace{i+1,\ldots,i+1}^{j}, \underbrace{i,\ldots i}_{r-j}) \qquad k = ri+j,\ 0 \leq j \leq r-1.$$

Christoffel-Darboux formula

Theorem (Daems and Kuijlaars)

Let $(\vec{n}_k)_{0 \leq k \leq N}$ *be a path in* $\mathbb{N}^r$ *starting from* $\vec{n}_0 = \vec{0}$ *and ending in* $\vec{n}_N = \vec{n}$ *(where* $N = |\vec{n}|$*), such that* $\vec{n}_{k+1} - \vec{n}_k = \vec{e}_i$ *for some* $1 \leq i \leq r$. *Then*

$$(x-y) \sum_{k=0}^{N-1} P_{\vec{n}_k}(x) Q_{\vec{n}_{k+1}}(y) = P_{\vec{n}}(x) Q_{\vec{n}}(y) - \sum_{j=1}^{r} a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x) Q_{\vec{n}+\vec{e}_j}(y).$$

Christoffel-Darboux formula

Theorem (Daems and Kuijlaars)

Let $(\vec{n}_k)_{0 \leq k \leq N}$ be a path in $\mathbb{N}^r$ starting from $\vec{n}_0 = \vec{0}$ and ending in $\vec{n}_N = \vec{n}$ (where $N = |\vec{n}|$), such that $\vec{n}_{k+1} - \vec{n}_k = \vec{e}_i$ for some $1 \leq i \leq r$. Then

$$(x-y) \sum_{k=0}^{N-1} P_{\vec{n}_k}(x) Q_{\vec{n}_{k+1}}(y) = P_{\vec{n}}(x) Q_{\vec{n}}(y) - \sum_{j=1}^{r} a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x) Q_{\vec{n}+\vec{e}_j}(y).$$

The sum depends only on the endpoints of the path in $\mathbb{N}^r$ and not on the path between these points.

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