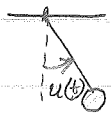


L1 §2 Initial Value Spaces

Initial value spaces are like phase spaces of mechanical systems.

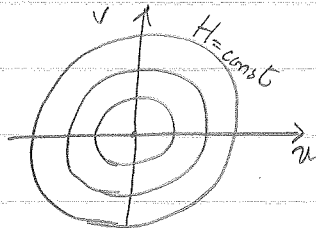
Consider^a simple pendulum, moving under influence of gravity:



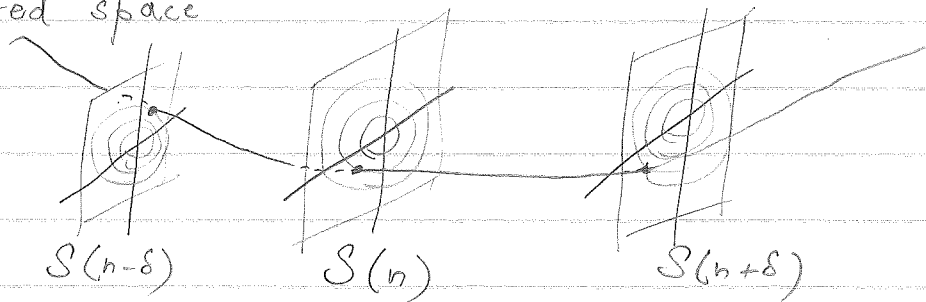
$$\begin{cases} \dot{u}(t) = v(t) \\ \dot{v}(t) = -u(t) \end{cases} \Rightarrow H = \frac{v^2}{2} + \frac{u^2}{2} \quad (2.1)$$

is conserved in time.

Knowledge of initial values $(u(0), v(0))$ determines H for all t , and therefore describes the motion qualitatively for all time.



For DPEs, we will see that solutions follow trajectories in a fibred space



The plane pendulum motion is actually governed by

$$\begin{cases} \dot{u}(t) = v(t) \\ \dot{v}(t) = -\sin(u(t)) \end{cases} \quad (2.2)$$

where the case $|u| \ll 1$ gives the limiting form (2.1).

This system also has conserved total energy

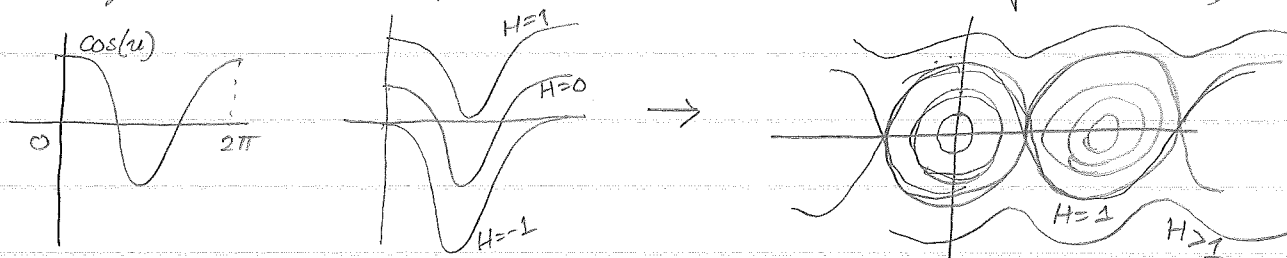
$$H(u, v) = \frac{v^2}{2} - \cos(u)$$

Now the motion parametrizes curves that are no longer circles.

L1 ②

$$u(0) = \frac{(2n+1)\pi}{2}, \quad v(0) = 0 \Rightarrow H = 0 \Rightarrow v = \pm \sqrt{2 \cos(u)}$$

$$u(0) = \pi, \quad v(0) = 0 \Rightarrow H = 1 \Rightarrow v = \pm \sqrt{2(1 + \cos(u))}$$



Consider the plane pendulum in complex variables. Let

$$x(t) = \exp(iu(t))$$

$$\Rightarrow \dot{x} = i v \cdot x \Rightarrow v = -i \frac{\dot{x}}{x}$$

$$\Rightarrow H = -\frac{\dot{x}^2}{x^2} - \frac{1}{2} \left(x + \frac{1}{x} \right)$$

or writing $\dot{x} = y$, we obtain

$$f(x, y) = y^2 + kx^2 + \frac{1}{2} (x^3 + x) = 0, \quad \text{where } H = k.$$

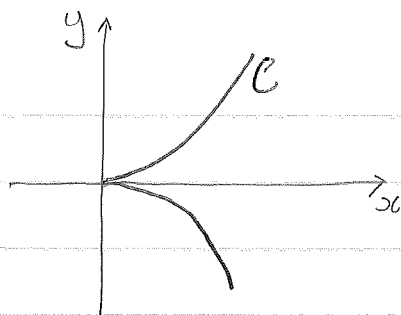
Note that all these curves intersect at $(x, y) = (0, 0)$, regardless of the value of $H = k$. This is called a base point.

This causes difficulties in the description of the motion $(x(t), y(t))$ of points on such curves. How do we distinguish the motion and continue the solutions through such points.

To explain the procedure that answers this question, we consider a simpler curve.

$$C: f(x, y) = y^2 - x^3 = 0,$$

which has a singularity at $(0, 0)$. I.e. both f and ∇f vanish there. Note: $\nabla f = (f_x, f_y) = (-2x^2, 2y)$.



We can resolve the flow of the parametrization of the curve through $(0,0)$ by taking a sequence of new coordinates. The first transformation is

$$(x, y) = (x_1, x_1, y_1) \Rightarrow x_1 = x, y_1 = y/x$$

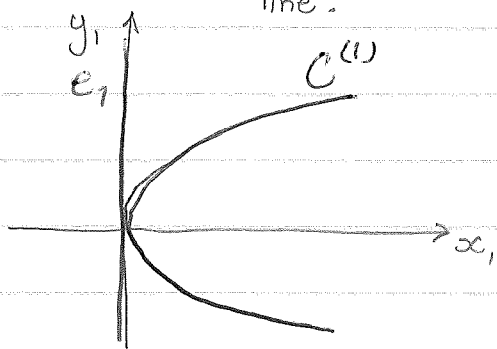
$\Rightarrow$

$$f(x, y) = x_1^2 y_1^2 - x_1^3 = x_1^2 (y_1^2 - x_1)$$

$e_1 = \{x_1 = 0\}$
called an exceptional line.

$f^{(1)} = y_1^2 - x_1$,
called the strict transform of f .

$\Rightarrow$



where $C^{(1)}: f^{(1)}(x_1, y_1) = 0$.

But e_1 and $C^{(1)}$ being tangential at $(0,0)$ still causes ambiguity for the parametrizing functions. So we resolve again:

$$(x_1, y_1) = (x_2 y_2, y_2) \Rightarrow x_2 = \frac{x_1}{y_1}, y_2 = y_1$$

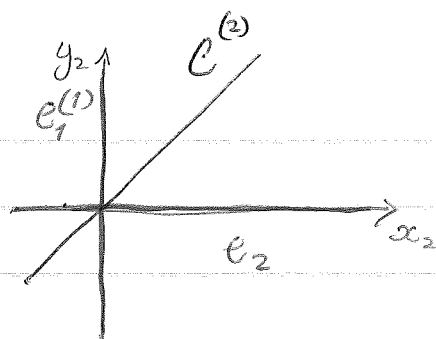
$$\Rightarrow f^{(1)}(x_1, y_1) = y_2^2 - x_2 y_2 = y_2 (y_2 - x_2)$$

$e_2 = \{y_2 = 0\}$

$f^{(2)} = y_2 - x_2$

and e_1 becomes $\{x_2 = 0\} = e_1^{(1)}$.

L_1 (4)



But $C^{(2)}$, $e_1^{(1)}$ and e_2 all intersect at $(x_2, y_2) = (0, 0)$.

This still causes a problem. (Note that e_1 is now $e_1 - e_2$ and no longer contains this point.)

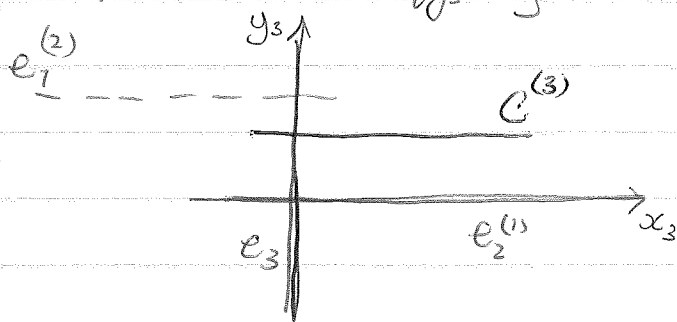
So we resolve again:

$$(x_2, y_2) = (x_3, x_3 y_3) \text{ or } (x_3, y_3) = (x_2, y_2/x_2)$$

$$\Rightarrow f^{(2)}(x_2, y_2) = x_3 y_3 - x_3 = x_3 (y_3 - 1)$$

$\downarrow$ $e_3: \{x_3=0\}$ $\downarrow$ $f^{(3)}$

Note: $e_2^{(1)}$ is $\{y_3=0\}$ but $e_1^{(1)}$ becomes $y_3=\infty$, no longer visible.



Now the curves only intersect pairwise and do so transversely. This is known as a "good resolution," and is guaranteed to exist by Hironaka's theorem in a field of any characteristic.

Each step above is called a "blow-up" and such operations were well known to Newton, who was studying Puiseux series expansions of y as a function of x around such singularities.

The same approach works to resolve a pencil of curves through its base points.

Note that any curve obtained as an exceptional line from a blow-up is a curve of self-intersection number that is one less than the curve we started from.

So e_1 has self-intersection -1

$$\begin{array}{ccccccc} e_1 - e_2 & u & & u & & -2, \\ \text{and } e_2 - e_3 & " & & " & & -2. \end{array}$$

Behind the space we have constructed lies a deeper algebraic structure. Consider the lines of self-intersection -2 :

$$\alpha_1 = e_1 - e_2$$

$$\alpha_2 = e_2 - e_3$$

and equip the equivalence classes of all lines with an intersection form $(\mid)$:

$$(e_i \mid e_j) = \begin{cases} -1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Note: $\left. \begin{array}{l} (\alpha_i \mid \alpha_i) = -2, \quad i=1,2 \\ \text{and } (\alpha_1 \mid \alpha_2) = +1 \end{array} \right\} \Rightarrow 4 \cos(\theta_{12}) = 1$
when interpreted as an inner product, with θ_{12} as angle between α_1, α_2 .

This leads to a reflection group of type A_2 .

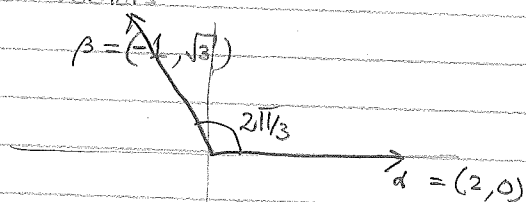
Every algebraic curve corresponds to a reflection group of type A-D-E.

(This is known as Du Val or McKay correspondence, and Arnold's classification of curves.)

When the "curves" are no longer autonomous, we obtain a classification of initial value spaces of Painlevé and discrete Painlevé equations.

L2

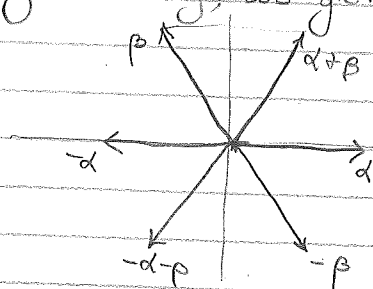
Consider two vectors



Reflecting β in α means reflecting it across a mirror orthogonal to α

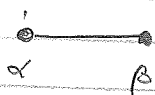
$$\begin{aligned}
 w_{\alpha}(\beta) &= \beta - 2 \frac{(\alpha, \beta)}{(\alpha, \alpha)} \alpha \\
 &= (-1, \sqrt{3}) - \frac{2(-2)}{4} (2, 0) \\
 &= (1, \sqrt{3}) \\
 &= \alpha + \beta
 \end{aligned}$$

Repeatedly reflecting, we get



a "root system" with 6 roots. It forms a group under reflection. Every element is an integer linear combination of $\alpha, \beta \Rightarrow$ these are "simple roots".

The group is named A_2 and represented by the Dynkin diagram



More generally, a reflection group or Weyl group has

- a root system $\alpha_1, \dots, \alpha_n$
- reflections $w_i(\alpha_j) = \alpha_j - 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \alpha_i$
- co-roots $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$
- weights $h_1, \dots, h_n$
s.t. $(h_i, \alpha_j^\vee) = \delta_{ij}$

satisfying the crystallographic property: $(\alpha_i, \alpha_j^\vee) \in \mathbb{Z}$

$$\Rightarrow (\alpha_i, \alpha_j^\vee) (\alpha_i, \alpha_j^\vee) = 4 \cos^2(\theta_{ij}) \in \mathbb{N}$$

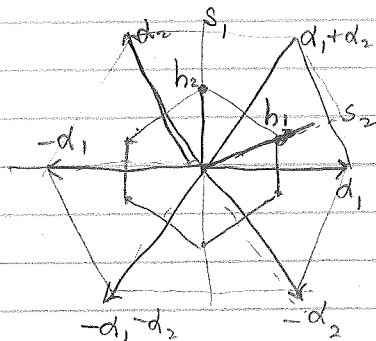
$$\Rightarrow \cos^2(\theta_{ij}) = \frac{n}{4}, \quad n \in \mathbb{N}$$

i.e. $\cos^2(\theta_{ij}) = 0, \frac{1}{4}, \frac{1}{2}$ or $\frac{3}{4}$

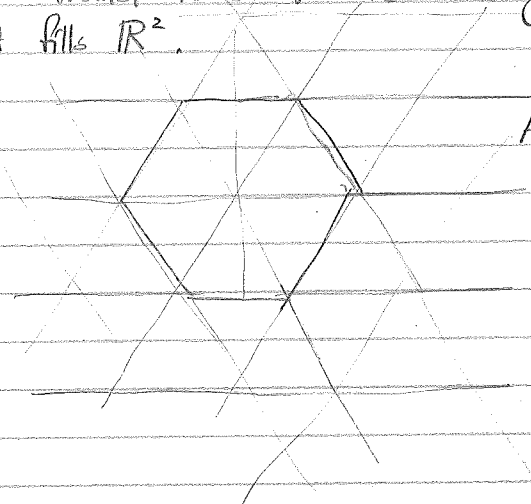
$\Rightarrow \cos(\theta_{ij}) = 0, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2}$

$\Rightarrow \theta_{ij} = \pi - \theta_{s_i, s_j} = \pi - \frac{\pi}{m_{ij}},$ with $m_{ij} = 2, 3, 4$ or 6 .

For A_2 :

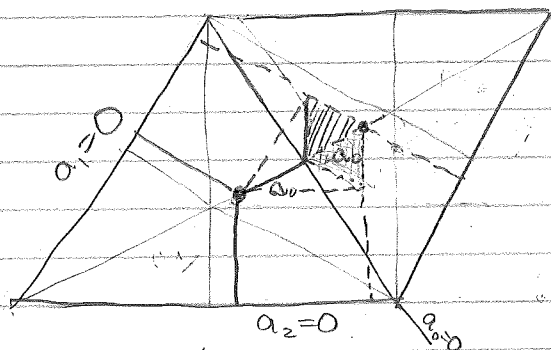


Now take translations of the root system to build an affine Weyl lattice that fills $\mathbb{R}^2$.



$A_2 \Rightarrow$ a triangular lattice $A_2^{(1)}$
affine reflections on the lattice give us an affine Weyl group also called $A_2^{(1)}$.
finite gp $\nearrow$ infinite gp

Now take coordinates, inside each triangle given by (a_0, a_1, a_2) .



Let s_j be the reflection across the side given by $a_j = k$, for some $k \in \mathbb{N}$.

Note: $a_0 + a_1 + a_2 = \text{const} = k$ say.
The shaded triangle is isosceles.

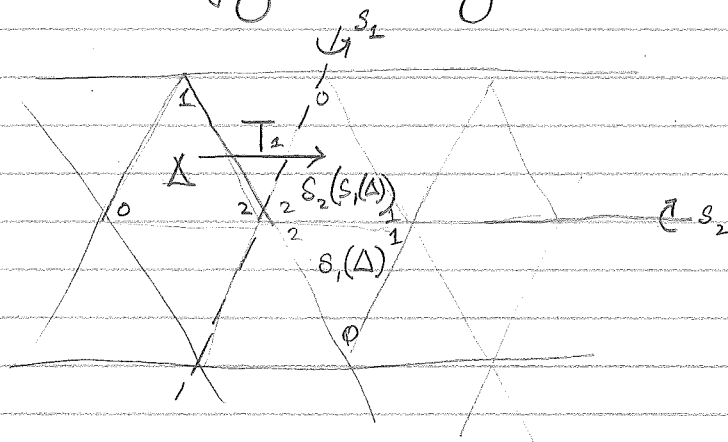
$\Rightarrow s_0(a_0, a_1, a_2) = (-a_0, a_1 + a_0, a_2 + a_0)$

Similarly

$s_1(a_0, a_1, a_2) = (a_0 + a_1, -a_1, a_2 + a_1)$

$s_2(a_0, a_1, a_2) = (a_0 + a_2, a_1 + a_2, -a_2)$

We want to develop dynamics on this lattice by taking a translation from one triangle to a conjugate triangle.



We need a permutation π to get back the original labelling.

$\Rightarrow$

$$\begin{aligned} T_1 &= \pi \circ s_2 \circ s_1 & \text{where } \pi(0, 1, 2) &= (1, 2, 0) \\ \text{Similarly } T_2 &= s_1 \circ \pi \circ s_2 & \text{and } T_1(a_0) &= a_0 + k \quad T_1(a_2) = a_2 \\ T_0 &= s_2 \circ s_1 \circ \pi & T_2(a_1) &= a_1 - k \end{aligned}$$

The extended affine Weyl (or Coxeter) group is $\tilde{W}(A_2^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle$ where

$$s_j^2 = 1, \quad (s_j s_{j+1})^3 = 1, \quad \pi s_j = s_{j+1} \pi \quad \text{where } j \in \mathbb{N} \text{ mod } 3$$

and $\pi^3 = 1$.

π is called a diagram automorphism (Noumi, 2004).

But behind this facade, we can interpret the roots as divisors in a space of equivalence classes of lines with an intersection form.

In this associated space, with coordinates (f_0, f_1, f_2) say, the action of s_j is given by

	f_0	f_1	f_2
s_0	f_0	$f_1 + \frac{a_0}{f_0}$	$f_2 - \frac{a_0}{f_2}$
s_1	$f_0 - \frac{a_1}{f_1}$	f_1	$f_2 + \frac{a_1}{f_2}$
s_2	$f_0 + \frac{a_2}{f_2}$	$f_1 - \frac{a_2}{f_1}$	f_2

with $f_0 + f_1 + f_2 = t$.

Letting $u_n = T_1^n(f_1)$, $v_n = T_1^n(f_0)$

we find

$$\begin{cases} u_n + u_{n+1} = t - v_n - \frac{a_0 + kn}{v_n} \\ v_n + v_{n+1} = t - u_n + \frac{a_1 - kn}{u_n} \end{cases}$$

which is the first discrete Painlevé equation or dP_1 . (Fokas, Its & Kitaev 1992)

The case $a_0 = 0$, $k = -2$, $a_1 = 1$ was discovered by Shohat (1939) in his extension of classical orthogonal polynomials. It was rediscovered in 1980 by Bessis et al in the study of combinatorics of closed graphs, and in 1991 by Douglas et al in quantum gravity (Hermitian RMT). It is now referred to as the string equation.

We know almost nothing about its generic (highly transcendental) solutions, except for $n \rightarrow \infty$.

Many discrete Painlevé eqns are now known.

I will give ^{as} a sketch of them:

- integrable systems
- how related to complex algebraic surfaces
- special features of solutions.

Integrability
Geometry
Asymptotics

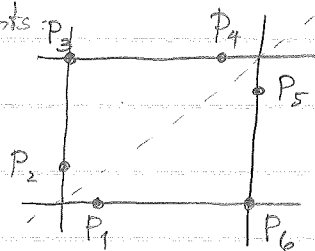
L_3 updated: §3 Geometry of Initial Value Spaces

Recall the pencil of curves (for given a, b)

$$f(x, y) = x^2 y^2 + bxy(x+y) + ab(x+y) + a - kxy = 0$$

$\uparrow$ free parameter

has base points p_i in $\mathbb{P}^1 \times \mathbb{P}^1$



There is a reflection symmetry in the pencil: $f(x, y) = f(y, x)$, under which $\{P_2, P_3, P_4\} \leftrightarrow \{P_1, P_6, P_5\}$

So we'll restrict our attention on blowing up P_1, P_6, P_5 .

To blow up a point $(x, y) = (\alpha, \beta)$, means taking new coords

$$\begin{cases} x_1 = \frac{x - \alpha}{y - \beta} \\ y_1 = y - \beta \end{cases} \quad \text{or} \quad \begin{cases} x_2 = x - \alpha \\ y_2 = \frac{y - \beta}{x - \alpha} \end{cases}$$

[This is the simplest transformation. In general, we can take any transfⁿ that is locally analytic and invertible s.t. a new variable takes form $0/0$ at the point.]

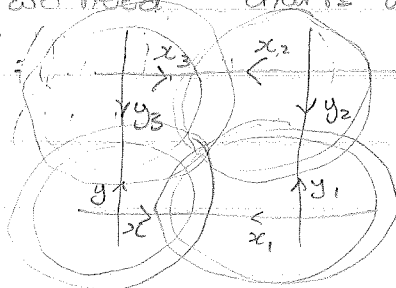
Note: ^{Embedding} $(x_1, \dots, x_n) \in \mathbb{C}^n$, in $\mathbb{P}^n$ means that we take $(x_1, \dots, x_n, 1)$

$$\underbrace{[x_1, \dots, x_n, 1]}_{\text{affine coordinates}} = \underbrace{\left[\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}, 1 \right]}_{\text{homogeneous coords}} = [x_1, \dots, x_n, x_{n+1}]$$

with at least one $x_i \neq 0$

In affine coords, we need $(n+1)$ charts to cover all domains in $\mathbb{P}^n$.

We took $\mathbb{P}^1 \times \mathbb{P}^1$:



Blowing-up P_1 :

$$\begin{cases} x_{11} = \frac{x + 1/b}{y} \Rightarrow x = -\frac{1}{b} + x_{11}y_{11} & \text{or} & \begin{cases} x_{12} = \frac{x + 1/b}{y} \Rightarrow x = -\frac{1}{b} + x_{12} \\ y_{12} = \frac{y}{x + 1/b} \Rightarrow y = x_{12}y_{12} \end{cases} \end{cases}$$

$$\Rightarrow f = \left(-\frac{1}{b} + x_{11}y_{11}\right)^2 y_{11}^2 + b \left(-\frac{1}{b} + x_{11}y_{11}\right) y_{11} \left(-\frac{1}{b} + x_{11}y_{11} + y_{11}\right) + ab \left(-\frac{1}{b} + x_{11}y_{11} + y_{11}\right) - k \left(-\frac{1}{b} + x_{11}y_{11}\right) y_{11}$$

$$= y_{11} \left\{ y_{11} \left(-\frac{1}{b} + x_{11}y_{11}\right)^2 + b \left(-\frac{1}{b} + x_{11}y_{11}\right) \left(-\frac{1}{b} + y_{11} + x_{11}y_{11}\right) + ab \left(1 + x_{11}\right) - k \left(-\frac{1}{b} + x_{11}y_{11}\right) \right\}$$

No new base pts. (Taking $-\frac{1}{b} + x_{11}y_{11} = 0 \Leftrightarrow x_{11} = 0 \rightarrow$ blowing back down to P_1 .)

Exercise: Write f in (x_{12}, y_{12}) to show $\exists$ no new base pt in that chart either.

We have an exceptional line $E_1 = \{y_{11} = 0\} \cup \{x_{12} = 0\}$.

Exercise: Show that no new base pts arise from blowing up P_5 .

Blowing-up P_6 :

$$\begin{cases} x_{61} = \frac{x_1}{y_1} \Rightarrow x_1 = x_{61}y_{61} & \text{or} & \begin{cases} x_{62} = \frac{x_1}{y_1} \Rightarrow x_1 = x_{62} \\ y_{62} = \frac{y_1}{x_1} \Rightarrow y_1 = x_{62}y_{62} \end{cases} \end{cases}$$

$$\begin{aligned} f_1 &= y_1^2 + b y_1 (1 + x_1 y_1) + ab (x_1 + x_1^2 y_1) + a x_1^2 - k x_1 y_1 \\ &= y_{61}^2 + b y_{61} (1 + x_{61} y_{61}^2) + ab (x_{61} y_{61} + x_{61}^2 y_{61}^3) + a x_{61}^2 y_{61}^2 - k x_{61} y_{61}^2 \\ &= y_{61} \left\{ y_{61} + b (1 + x_{61} y_{61}^2) + ab x_{61} (1 + x_{61} y_{61}^2) + a x_{61}^2 y_{61} - k x_{61} y_{61} \right\} \\ &=: y_{61} f_1^{(2)} \end{aligned}$$

$\Rightarrow$ New base pt on exceptional line $E_6 = \{y_{61} = 0\}$

given by $b + ab x_{61} = 0 \Rightarrow P_8 = (x_{61}, y_{61}) = \left(-\frac{1}{a}, 0\right)$

Exercise: Carry out the blow-up in chart 62 and show P_8 appears

there on $E_6 = \{x_{62} = 0\}$ as $(x_{62}, y_{62}) = (0, -a)$.

$$\text{Here } f_1^{(1)} = x_{62} y_{62}^2 + b y_{62} (1 + x_{62}^2 y_{62}) + ab (1 + x_{62}^2 y_{62}) + a x_{62} - k x_{62} y_{62}$$

Blowing-up P_8 :

$$\begin{cases} x_{81} = \frac{x_{62}}{y_{62}+a} \Rightarrow x_{62} = x_{81}y_{81} & \text{or} \begin{cases} x_{82} = x_{62} \Rightarrow x_{62} = x_{82} \\ y_{82} = \frac{y_{62}+a}{x_{62}} \end{cases} \\ y_{81} = y_{62}+a & y_{62} = -a+y_{81} \end{cases}$$

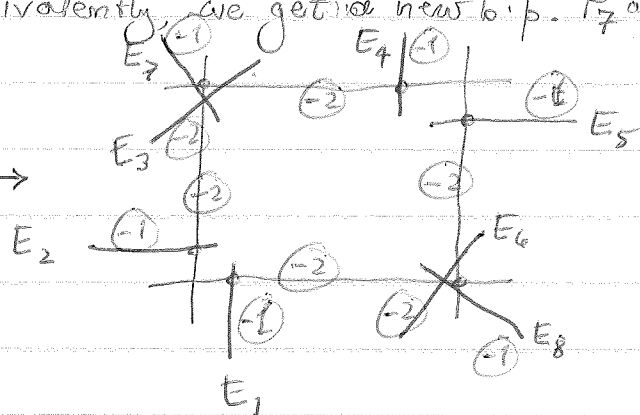
Now

$$f_1^{(1)} = y_{81} \left\{ x_{81}(-a+y_{81})^2 + b + b x_{81}^2 y_{81}(-a+y_{81}) + a b x_{81}^2 y_{81}(-a+y_{81}) + a x_{81} - k x_{81}(-a+y_{81}) \right\}$$

No new basepts (because coeff of k doesn't vanish
 $\Rightarrow E_8 = \{y_{81} = 0\}$.)

Equivalently we get a new b.p. P_7 after blowing up P_3 .

Call this resolved space $X \rightarrow$



$$\begin{aligned} \text{Let } H_1 &= \{x = \text{const}\} \\ &\cup \{x_{61} = \text{const}\} \\ H_2 &= \{y = \text{const}\} \\ &\cup \{y_{82} = \text{const}\} \end{aligned}$$

The curves of self-intersection (-2) are

$$D_0 = E_6 - E_8$$

$$D_1 = H_2 - E_1 - E_6$$

$$D_2 = H_1 - E_2 - E_3$$

$$D_3 = E_3 - E_7$$

$$D_4 = H_2 - E_3 - E_4$$

$$D_5 = H_1 - E_5 - E_6$$

We use, the bilinear intersection form $(\quad | \quad)$ where

$$(E_i | E_j) = 0 \quad \text{if } i \neq j$$

$$(E_i | E_i) = -1$$

$$(H_j | E_i) = 0 \quad \forall i, j$$

$$(H_i | H_j) = 0$$

$$(H_1 | H_2) = 1$$

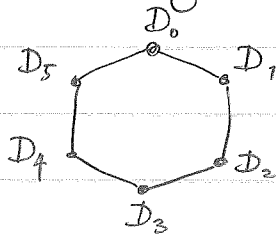
Note: $(D_0 | D_1) = (E_6 - E_8 | H_2 - E_1 - E_6)$

$$= \underbrace{(E_6 | H_2)}_0 - \underbrace{(E_6 | E_1)}_0 - \underbrace{(E_6 | E_6)}_{-1} - \underbrace{(E_8 | H_2)}_0 + \underbrace{(E_8 | E_1)}_0 + \underbrace{(E_8 | E_6)}_0$$

$$= 1$$

Exercise: Calculate $(D_i | D_j)$ for $i, j = 0, \dots, 5$.

Connecting D_i and D_j only if $(D_i | D_j) \neq 0$, we get



i.e. Dynkin diagram of type $A_5^{(1)}$.
with D_j considered as simple roots
 $\delta = D_0 + D_1 + \dots + D_5$ is called the "null-root".

In the next lecture, I will show how to deduce a discrete Painlevé equation from the symmetries of this initial-value space.
For this, we need to deduce the symmetry group.

To understand this, consider the equivalence classes of all lines in X : $\langle H_1, H_2, E_1, \dots, E_8 \rangle = \text{Pic}(X)$.

$\text{Pic}(X)$ has 10-dims, while $\langle D_0, \dots, D_5 \rangle$ is only 6-dim $\mathbb{R}$.

The orthogonal subspace of $\text{Pic}(X)$, orthog^l to $\langle D_0, \dots, D_5 \rangle$, is the symmetry group.

So we look for $F = \mu_1 H_1 + \mu_2 H_2 + \nu_1 E_1 + \dots + \nu_8 E_8$

s.t. $(F | D_j) = 0 \quad \forall j \in \{0, \dots, 5\}$.

First₁ generalize slightly to $P_1, \dots, P_8$ given by

$$P_1 = (-a_0^{-1}, 0)$$

$$P_5 = (\infty, -a_1^{-1})$$

$$P_2 = (0, -a_1)$$

$$P_6 = (\infty, 0)$$

$$P_3 = (0, \infty)$$

$$P_7 = (0, \infty) \text{ s.t. } xy = -c^2 a_0 a_1^2$$

$$P_4 = (-a_0, \infty)$$

$$P_8 = (\infty, 0) \text{ s.t. } xy = -c^2 a_0 a_1$$

(Same as above if $a_0 = b$, $a_1 = 1/b$.)

The story so far:

- Given $f(x, y) = 0$, we resolved the base pts $P_i, i=1, \dots, 8, \mapsto E_i, i=1, \dots, 8$.
- Curves of self-intersection -2 were identified $\Rightarrow D_j, j=0, \dots, 5$.
- Defined intersection form $(\cdot, \cdot)$
- Identified $\{D_j\}$ as ^{simple} roots of $A_5^{(1)}$
- Defined $\text{Pic}(X) = \langle H_1, H_2, E_1, \dots, E_8 \rangle$ where
 $H_1 = \{y = \text{const}\}, H_2 = \{x = \text{const}\}.$

In this lecture: (J. Nakazono & Shi, Reflection groups & discrete integrable systems, J. Integrable Systems (2016) 00 1-37)

- We identify the symmetry group
- Define reflections
- Deduce actions of reflections on parameters and x, y .
- Construct discrete Painlevé eqn.

Conclusion:

L_0 : History (Shohat);
 $P_{\text{IV}} \xrightarrow{\text{BTs}} dP_{\text{II}}$
 BTs $\leftrightarrow$ reflection groups
 Outline of Sakai's description

L_3 : Geometry of IVS
 Resolution of base pts
 Identifying resolved space
 with an affine Root system.
 Intersection form.

L_1 : Initial value spaces
 Curves (from Hamiltonians)
 Resolution of singularities

L_4 : Constructing discrete Painlevé eqns

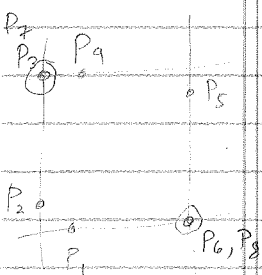
L_2 : Reflection groups
 Translations on affine Weyl lattices
 Actions on coordinates in lattice
 Birational actions on associated space

ⓐ: How does this theory
 carry over to
 orthogonal polynomials?
 Are OPs related to affine Weyl gps?
 Is there a classification
 like Sakai's for all
 possible OPs?

L4

space X

Consider a slightly more general setting with the same group. (gives prev case if $a_0 = b, a_1 = 1/b$)
With $P_1, \dots, P_8$ given by



$$P_1 = (-a_0^{-1}, 0)$$

$$P_2 = (0, -a_1)$$

$$P_3 = (0, \infty)$$

$$P_4 = (-a_0, \infty)$$

$$P_5 = (\infty, -a_1^{-1})$$

$$P_6 = (\infty, 0)$$

$$P_7 = (0, \infty) \text{ s.t. } xy = -c^2 a_0 a_1 a_2^2$$

$$P_8 = (\infty, 0) \text{ s.t. } xy = -c^2 a_0 a_1$$

$\Rightarrow D_0, \dots, D_5$ forming an $A_5^{(1)}$ root-system with "null root"
 $\delta = D_0 + D_1 + \dots + D_5$
 $= 2H_1 + 2H_2 - E_1 - E_2 - E_3 - \dots - E_8$

$$D_0 = E_6 - E_8$$

$$D_1 = H_2 - E_1 - E_6$$

$$D_2 = H_1 - E_2 - E_3$$

$$D_3 = E_3 - E_7$$

$$D_4 = H_2 - E_2 - E_4$$

$$D_5 = H_1 - E_5 - E_6$$

In the Picard group generated by equivalence classes of lines $H_1, H_2, E_1, \dots, E_8$ this group is orthogonal to two other orthogonal subgroups

$$\begin{cases} \alpha_0 = H_1 - E_1 - E_4 \\ \alpha_1 = H_2 - E_2 - E_5 \\ \alpha_2 = H_1 + H_2 - E_3 - E_6 - E_7 - E_8 \end{cases} \text{ and } \begin{cases} \beta_0 = H_1 + H_2 - E_2 - E_4 - E_6 - E_8 \\ \beta_1 = H_1 + H_2 - E_1 - E_3 - E_5 - E_7 \end{cases}$$

$$\Downarrow A_2^{(1)}$$

$$\text{Check: e.g. } (D_0/\alpha_0) = 0$$

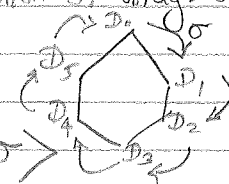
$$j=1 \Rightarrow (H_1/H_2) + (E_1/E_1) = 1-1=0$$

$$\Downarrow A_1^{(1)}$$

So the initial value space X is unchanged under the action of $(A_2 + A_1)^{(1)}$ which forms a symmetry group. Also under the action of diag. automorphism

Consider the extended affine Weyl group

$$\tilde{W}((A_2 + A_1)^{(1)}) = \langle s_0, s_1, s_2, w_0, w_1, \sigma \rangle$$



Here the action of s_i is reflection by α_i
 w_j " " " " by β_j

Convention

$$f.w = w^t.f = \bar{f}$$

Each reflection is given by the intersection form: take any elt v in $P_d(X)$

$$v.s_i = v - 2 \frac{(v|\alpha_i)}{(\alpha_i|\alpha_i)} \alpha_i$$

$$v.w_j = v - 2 \frac{(v|\beta_j)}{(\beta_j|\beta_j)} \beta_j$$

We can work out the effects of these actions on the parameters $\{a_0, a_1, a_2, c\}$ and the coordinates x, y .

E.g. take $s_0(x) = \bar{x}$, $s_0(y) = \bar{y}$, $s(a_i) = \bar{a}_i$, $\bar{c} = s_0(c)$.

Then noting

$$(\alpha_0|\alpha_0) = (H_1 - E_1 - E_4 | H_1 - E_1 - E_4) = -1 - 1 = -2,$$

and

$$(H_1|\alpha_0) = 0, (H_2|\alpha_0) = 1, (E_j|\alpha_0) = 0 \text{ for } j \neq 1, 4.$$

$$H_2.s_0 = H_2 - 2 \frac{(H_2|\alpha_0)}{(\alpha_0|\alpha_0)} \alpha_0$$

we get

$$= H_2 - 2 \frac{(H_2|H_1 - E_1 - E_4)}{(-2)} (H_1 - E_1 - E_4)$$

$$= H_2 + H_1 - E_1 - E_4$$

$$H_1.s_0 = \bar{H}_1 \quad \leftarrow \text{notation for image of action}$$

$$\text{and } E_j.s_0 = \bar{E}_j$$

$$H_2.s_0 = \bar{H}_2 + \bar{H}_1 - \bar{E}_1 - \bar{E}_4$$

which means the lines $\{y = \text{const}\}$ are mapped birationally to $\{\bar{y} = \text{const}\}$

$$mH_1 + nH_2$$

$$- \sum_{i=1}^4 \mu_i E_i$$

is a curve of bi-deg (m, n) in $P^1 \times P^1$ passing through base pts

P_i with mult. μ_i .

i.e.

$$\bar{x} (A_1 x + A_2) + A_3 x + A_4 = 0$$

and similarly the $\deg(0, 1)$ lines $\{x = \text{const}\}$ are mapped to lines $\deg(1, 1)$ through P_1 and P_4 .

i.e.

$$\bar{y} (B_1 y(x + a_0) + B_2(x + a_0^{-1})) + B_3 y(x + a_0) + B_4(x + a_0^{-1}) = 0$$

from P_7 from P_1 from P_4 from P_7

$E_2 = (\text{from } P_2 = (0, -a_1))$ stays unchanged.

$$P_2 = (0, -a_1):$$

$$\bar{y}(B_1 a_0 a_1 + B_2 a_0^{-1} + B_3 a_1 n_0 + B_4 a_0^{-1}) = 0$$

$$P_3 = (0, \infty)$$

$$\bar{y}(B_1 a_0 + B_3 a_0^{-1}) = 0$$

$$\Rightarrow x=0 \text{ maps to } \bar{x}=0 \Rightarrow A_4=0$$

and so on for E_2, E_3, E_5 and E_6 :

$$\Rightarrow \text{we find } B_1 = B_4 = 0 \text{ while } B_2 = -a_0 B_3, \boxed{\bar{a}_1 = a_0 a_1.}$$

and from E_7, E_8 :

$$\bar{x} = \frac{-A_2 x}{A_2}, \quad \bar{y} = \frac{y(x + a_0)}{a_0(x + a_0^{-1})} = \frac{y(a_0 + x)}{(1 + a_0 x)}$$

$$\Rightarrow \bar{x} \bar{y} = \frac{-A_3}{A_2} \frac{(a_0 + x)}{(1 + a_0 x)} x y$$

$$\sim \frac{-A_3}{A_2} \cdot \frac{-c^2 a_0 a_1}{a_0} \quad \text{at } P_8$$

$$\Rightarrow -\bar{c}^2 \bar{a}_0 \bar{a}_1 = +c^2 a_1 \frac{A_3}{A_2}$$

i.e. $\frac{A_3}{A_2} = \frac{-\bar{c}^2 \bar{a}_0 a_0}{c^2}$

while at P_7 , we get

$$-\bar{c}^2 \bar{a}_0 \bar{a}_1 \bar{a}_2^2 = \frac{-A_3 a_0}{A_2} \cdot -c^2 a_0 a_1 a_2^2$$

$$\Rightarrow \frac{A_3}{A_2} = -\frac{\bar{c}^2}{c^2} \frac{\bar{a}_0 a_0 a_1 \bar{a}_2^2}{a_0^2 a_1 a_2^2} = -\frac{\bar{c}^2}{c^2} \bar{a}_0 a_0$$

$$\Rightarrow \bar{a}_2^2 = a_0^2 a_2^2 \quad \text{We take } \boxed{\bar{a}_2 = a_0 a_2}$$

Now consider

$$E_1 \cdot s_0 = \bar{E}_1 + \bar{H}_1 - \bar{E}_1 - \bar{E}_4 = \bar{H}_1 - \bar{E}_4 \Rightarrow P_1 \text{ is mapped to } P_4$$

$$E_4 \cdot s_0 = \bar{H}_1 - \bar{E}_1 \quad (\text{check!}) \Rightarrow P_4 \text{ is mapped to } P_1$$

$$E_1 \Rightarrow -\bar{a}_0 = -\frac{A_3}{A_2} \cdot \frac{-1}{a_0} \quad \text{and} \quad -\frac{1}{\bar{a}_0} = -\frac{A_3}{A_2} \cdot -a_0$$

$$\Rightarrow \bar{a}_0^2 = \frac{1}{a_0^2} \Rightarrow \boxed{\bar{a}_0 = \frac{1}{a_0}} \Rightarrow \frac{A_3}{A_2} = -1$$

and

$$-\bar{a}_0 = \frac{\bar{c}^2 \bar{a}_0 a_0}{c^2} \times \frac{-1}{a_0} = -\frac{\bar{c}^2}{c^2} \bar{a}_0 \Rightarrow \bar{c} = c$$

So far we have shown

$$s_0: (a_0, a_1, a_2, c) \mapsto \left(\frac{1}{a_0}, a_0 a_1, a_0 a_2, c \right)$$

and $s_0(x) = x, \quad s_0(y) = \frac{y(a_0 + x)}{1 + a_0 x}$

Note: $\bar{a}_0 \bar{a}_1 \bar{a}_2 = \frac{1}{a_0} \cdot a_0 a_1 \cdot a_0 a_2 = a_0 a_1 a_2 =: q \text{ const.}$

These are the building blocks that lead to a discrete Painlevé equation.
Taking $\pi = \sigma^2, \quad r = \sigma^3$ and defining translations on $\tilde{W}((A_2 + A_1)^{(1)})$ by

$$T_1 = \pi s_2 s_1$$

$$T_2 = \pi s_0 s_2$$

$$T_3 = \pi s_1 s_0$$

$$T_4 = \tau w_0$$

we get under repeated composition of T_1 : $F_n = T_1^n(x)$, $G_n = T_1^n(y)$

$$\begin{cases} G_{n+1} G_n = \frac{qc^2}{F_n} \frac{(1 + q^n a_0 F_n)}{(q^n a_0 + F_n)} \\ F_{n+1} F_n = \frac{qc^2}{G_{n+1}} \frac{(1 + q^n a_0 a_2 G_{n+1})}{(q^n a_0 a_2 + G_{n+1})} \end{cases}$$

which is a q -discrete Painlevé equation.

(See JNS 2016 J. Integ. Systems)

Similar for T_2, T_3 .

But T_4 leads to q - P_{IV}

and half-translation $R_1 = \pi^2 s_1$, where $R_1^2 = T_1$, leads to q - P_{II} .