Properties of orthogonal polynomials

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Outline

1. Orthogonal polynomials
2. Properties of classical orthogonal polynomials
3. Quasi-orthogonality and semiclassical orthogonal polynomials
4. The hypergeometric function
5. Convergence of Padé approximants for a hypergeometric function
    - Padé approximation
    - Padé approximant for ${}_2F_1(a, 1; c; z)$
    - Poles of the Padé approximant
    - Convergence in the Padé table

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We write $f(z) \equiv 0$ if $a_j = 0 \; \forall j \geq 0$.

The Padé Approximant

The $[m/n]$ Padé approximant for a formal power series

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is a rational function

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as $z \to 0$.

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The name comes from Henri Eugene Padé, a student of Hermite, who completed his thesis in 1892, but the approximant goes back to Cauchy and Jacobi.

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Let $f(z)$ be a formal power series.

Then $\forall m, n \geq 0$,

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Further, if after cancelling common factors in P and Q, we obtain

$$[m/n] = \hat{P}/\hat{Q},$$

then $\hat{Q}(0) \neq 0$ *and*

$$f(z) - [m/n](z) = O(z^{m+n+1-\ell}),$$

where

$$\ell = \min\left\{n - deg\ \hat{Q},\ m - deg\ \hat{P}\right\}.$$

Existence

Writing

$$P(z) = \sum_{j=0}^{m} p_j z^j,$$

$$Q(z) = \sum_{j=0}^{n} q_j z^j,$$

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the condition

$$(fQ - P)(z) = O(z^{m+n+1})$$

becomes

$$\left(\sum_{j=0}^{\infty} a_j z^j\right)\left(\sum_{k=0}^{n} q_k z^k\right) - \sum_{j=0}^{m} p_j z^j = O(z^{m+n+1}).$$

Consider the product of the two series

$$\left(\sum_{j=0}^{\infty} a_j z^j\right)\left(\sum_{k=0}^{n} q_k z^k\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{n} a_j\, q_k\, z^{j+k}.$$

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Introduce new indices of summation s and t by $k = s$ and $j = t - s$. Then $k + j = t$ and, since the old indices are restricted by $j \geq 0$ and $0 \leq k \leq n$, we have $0 \leq s \leq n$ and $t - s \geq 0$, i.e. $s \leq t$.

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It follows that summation over t runs from 0 to ∞ while summation over s runs from 0 to $\min(t, n)$ and hence

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or equivalently, changing back to dummy indices k and j

$$= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\min(n,j)} a_{j-k}\; q_k\right) z^j.$$

Now we can write $(fQ - P)(z) = O(z^{m+n+1})$ as

$$\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\min\{n,j\}} q_k \; a_{j-k} \right) z^j - \sum_{j=0}^{m} p_j \, z^j = O(z^{m+n+1}) \tag{1}$$

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Recalling that the coefficients of $z^0, \; z, \; z^2, \ldots, z^{m+n}$ are all zero on RHS of (1), we have

$$\begin{cases} \sum_{k=0}^{\min(j,\,n)} q_k \, a_{j-k} - p_j = 0, & j = 0, 1, 2, \ldots, m \\ \sum_{k=0}^{\min(j,\,n)} q_k \; a_{j-k} = 0, & j = m+1, \; m+2, \ldots, m+n \end{cases} \tag{2}$$

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(2) is a system of $(m+n+1)$ **homogeneous linear equations** in the $(m+n+2)$ variables $p_0, \; p_1, \ldots, p_m, \; q_0, \; q_1, \ldots, q_n$.

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Note that if $Q \equiv 0$ is such a solution, i.e. if $q_0 = q_1 = \ldots = q_n = 0$, then the first equation in (2) yields $p_j = 0,\ j = 0, 1, 2, \ldots, m$, which is a contradiction.

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Thus $Q \neq 0$ and [m/n] exists.

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$$(fQ - P)(z) = O(z^{m+n+1}) \tag{3}$$

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Then (3) $\times Q_1(z) -$ (4) $\times Q(z)$:

$$P_1(z)Q(z) - P(z)Q_1(z) = O(z^{m+n+1}). \quad (5)$$

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Then (3) $\times Q_1(z) -$ (4) $\times$ $Q(z)$:

$$P_1(z)Q(z) - P(z)Q_1(z) = O(z^{m+n+1}). \tag{5}$$

Since $P_1(z)Q(z) - P(z)Q_1(z)$ is a polynomial of degree $\leq m + n$, and (5) tells us that this polynomial has a zero at the origin of order $(m + n + 1)$, it follows that

$$P_1(z)Q(z) - P(z)Q_1(z) \equiv 0$$

or

$$P(z)/Q(z) \equiv P_1(z)/Q_1(z),$$

so $[m/n](z)$ is unique.

Consider $[m/n] = P/Q$ and assume that we can write, for some non-negative integer r with $r \leq m$ and n, and some polynomial S, that

$$P(z) = z^r\, S(t)\hat{P}(z)$$
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Since

$$z^r S(z)((f\hat{Q}(z) - \hat{P}(z))) = (fQ - P)(z) = O(z^{m+n+1}),$$

we can multiply by $\frac{1}{S(z)}$ since $S(0) \neq 0$ to deduce $z^r(f\hat{Q} - \hat{P})(z) = O(z^{m+n+1})$ which means

$$(f\hat{Q} - \hat{P})(z) = O(z^{m+n+1-r}) \tag{7}$$

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So $\hat{Q}(0) \neq 0$.

Thus we can multiply (10) by $1/\hat{Q}(z)$ to obtain

$$f(z) - \frac{\hat{P}(z)}{\hat{Q}(z)} = 0(z^{m+n+1-r}).$$

Finally, from (6),

$$\begin{aligned} m &\geq \deg\,(P) = r + \deg(S) + \deg(\hat{P}) \\ &\geq r + \deg\,(\hat{P}), \text{ so that} \end{aligned}$$

$r \leq m - \deg\,(\hat{P})$. Similarly, $r \leq n - \deg(\hat{Q})$, so if $\ell = \min\left\{n - \deg(\hat{Q}),\, m - \deg(\hat{P})\right\}$, we have $r \leq \ell$. Hence

$$m + n + 1 - r \geq m + n + 1 - \ell$$

and

$$f(z) - [m/n](z) = O(z^{m+n+1-\ell}).$$

Padé table

The $[m/n]$ Padé approximants for f can be arranged to form the Padé table of f.

$$\begin{array}{cccccc}
[0/0] & [0/1] & [0/2] & [0/3] & [0/4] & \dots \\
[1/0] & [1/1] & [1/2] & [1/3] & [1/4] & \dots \\
[2/0] & [2/1] & [2/2] & [2/3] & [2/4] & \dots \\
[3/0] & [3/1] & [3/2] & [3/3] & [3/4] & \dots \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Notice that the first column of the table is the sequence of partial sums of $f(z) = \sum_{j=0}^{\infty} a_j z^j$.

$$[m/0] = \sum_{j=0}^{m} a_j z^j$$

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The Padé table is normal if each entry in the table is normal.

Structure of the Padé table

The Padé table has a special structure.

Theorem (Padé)

The Padé table of a formal power series $f(z)$ consists of square blocks of size r, $1 \leq r \leq \infty$, for which

(a) *All elements in a square block are identical;*

(b) *No other entries in the Padé table of f are the same as elements in this block;*

(c) *If $[\hat{m}/\hat{n}] = \hat{P}/\hat{Q}$ is the top left hand corner of a square block, then $deg(\hat{P}) = \hat{m}$, $deg(\hat{Q}) = \hat{n}$, $Q(0) \neq 0$ and if $r = \infty$,*

$$f(z) - [\hat{m}/\hat{n}](z) \equiv 0$$

i.e. f is a rational function.

Padé approximant for ${}_2F_1(a, 1; c; z)$

Theorem (Padé, 1907)

Let $c \notin \mathbb{Z}^-$ and let $m \geq n-1$. Then the denominator polynomial in the $[m/n]$ Padé approximant $P_{mn}(z)/Q_{mn}(z)$ for ${}_2F_1(a, 1; c; z)$ is given by

$$Q_{mn}(z) = {}_2F_1(-n, -a-m; -c-m-n+1; z)$$

and

$$\begin{aligned} R_{mn}(z) &= Q_{mn}(z)\ {}_2F_1(a, 1; c; z) - P_{mn}(z) \\ &= S_{mn}\ z^{m+n+1}\ {}_2F_1(a+m+1, n+1; c+m+n+1; z) \end{aligned}$$

where

$$S_{mn} = n! \frac{(a)_{m+1}(c-a)_n}{(c)_{m+n}(c+m)_{n+1}}$$

$$
\begin{array}{ccccccc}
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[1/0] & [1/1] & [1/2] & [1/3] & [1/4] & [1/5] & \dots \\
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Theorem (van Rossum, 1955)

If $a, c, c - a \notin \mathbb{Z}^-$, the Padé approximants for ${}_2F_1(a, 1; c; z)$ are normal for $m \geq n - 1$.

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Theorem (de Bruin, 1976)

The Padé table for the hypergeometric series ${}_2F_1(a, 1; c; z)$ with $c > a > 0$ is normal.

The numerator polynomial

We do not have a closed form for the numerator polynomial of the Padé approximants for ${}_2F_1(a, 1; c; z)$ so we do not know where the zeros of the approximant lie. The numerator polynomial $P_{mn}(z)$ is determined by

$$f(z)Q_{mn}(z) - P_{mn}(z) = 0(z^{m+n+1})$$

since $Q_{mn}(z)$ is known.

The numerator polynomial

We do not have a closed form for the numerator polynomial of the Padé approximants for ${}_2F_1(a, 1; c; z)$ so we do not know where the zeros of the approximant lie. The numerator polynomial $P_{mn}(z)$ is determined by

$$f(z)Q_{mn}(z) - P_{mn}(z) = 0(z^{m+n+1})$$

since $Q_{mn}(z)$ is known.

For $m = n - 1$, $P_{mn}(z)$ is the polynomial we obtain from the first $(m + 1)$ terms in the product

$${}_2F_1(a, 1; c; z) \; {}_2F_1(-n, -a - m; -c - m - n + 1; z)$$

The numerator polynomial

Thus

$$P_{mn}(z) = \sum_{r=0}^{m} \sum_{l=0}^{r} \frac{(a)_{r-l}(-n)_l(-a-m)_l}{(-c-m-n+1)_l(c)_{r-l}l!}\ z^r$$

for $0 \leq r \leq m$.

The numerator polynomial

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for $0 \le r \le m$.

Example

For $a = 2,\ c = 6,\ m = 3$ and $n = 4$,

$$P_{34}(z) = 1 - \frac{4}{3}z + \frac{344}{693}z^2 - \frac{1}{22}z^3$$

which is not equal to ${}_2F_1(-3, \alpha; \beta; z)$ for any α, β.

Poles of the Padé approximant for ${}_2F_1(a, 1; c; z)$

Corollary

For $c \notin \mathbb{Z}^-$ and $m \geq n-1$, the poles of the $[m/n]$ Padé approximant for ${}_2F_1(a, 1; c; z)$ lie in the intervals

(i) $(0, 1)$ *if* $a < c < 1 - m - n$

(ii) $(1, \infty)$ *if* $c > a > n - m - 1$

(iii) $(-\infty, 0)$ *if* $a > n - m - 1$ *and* $c < 1 - m - n$.

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Remark

(ii) If $m \geq n - 1$ and $c > a > 0$ we have normality in the Padé table and the poles of the Padé approximant lie on the cut $(1, \infty)$

The poles of the Padé approximant and convergence in the table

- The location and behavior of the zeros and poles of Padé approximants for various special functions, as well as the asymptotic zero and pole distribution, has been studied by many authors, most notably E. Saff and R. Varga [1978]
- The convergence of different types of sequences in the Padé table has been studied extensively.
  - Exponential function [Perron, 1957]
  - ${}_1F_1(1; c; z)$ with $c \notin \mathbb{Z}^-$ [de Bruin, 1976]

Lemma

For $m \geq n-1$ and $c > a > 0$ we have that

$$\begin{aligned} R_{mn}(z) &= Q_{mn}(z){}_2F_1(a, 1; c; z) - P_{mn}(z) \\ &= S_{mn}\ z^{m+n+1}\ {}_2F_1(a+m+1, n+1; c+m+n+1; z) \end{aligned}$$

tends to zero uniformly in z as $m \to \infty$, $n/m \to \rho$ with $0 < \rho \leq 1$ on compact subsets of $|z| < 1$.

The Padé table

$$
\begin{array}{ccccccc}
[0/0] & [0/1] & [0/2] & [0/3] & [0/4] & [0/5] & \ldots \\
[1/0] & [1/1] & [1/2] & [1/3] & [1/4] & [1/5] & \ldots \\
[2/0] & [2/1] & [2/2] & [2/3] & [2/4] & [2/5] & \ldots \\
[3/0] & [3/1] & [3/2] & [3/3] & [3/4] & [3/5] & \ldots \\
[4/0] & [4/1] & [4/2] & [4/3] & [4/4] & [4/5] & \ldots \\
[5/0] & [5/1] & [5/2] & [5/3] & [5/4] & [5/5] & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Convergence in the Padé table for ${}_2F_1(a, 1; c; z)$, $c > a > 0$

Theorem

Let $a, c, c - a \notin \mathbb{Z}^-$ and $m \geq n - 1$. For $c > a > 0$, the sequence of $[m/n]$ Padé approximants

$$\frac{P_{mn}(z)}{Q_{mn}(z)}$$

converges to

$${}_2F_1(a, 1; c; z)$$

for $m \to \infty$, $n/m \to \rho$ with $0 < \rho \leq 1$, uniformly in z on compact subsets of $|z| < 1$.