Properties of orthogonal polynomials

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Outline

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where the parameters a, b, c and z may be real or complex and

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The infinite series converges for $|z| < 1$ (see Assignment 2, Exercise 1) and this radius of convergence can be extended by analytic continuation, so that $_2F_1$ is a single valued analytic function of z on $\mathbb{C}_{[1,\infty)}$.

Lemma

For $k, n \in \mathbb{N}$*,*

$$(-n)_k = \begin{cases} (-1)^k \dfrac{n!}{(n-k)!} & \textit{for } 0 \le k \le n \\ 0 & \textit{for } k \ge n+1. \end{cases}$$

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When a (or b) is a negative integer, say $a = -n$, the series terminates

$$_2F_1(-n, b; c; z) = 1 + \sum_{k=1}^{n} \frac{(a)_k (b)_k z^k}{(c)_k k!}, \quad |z| < 1,$$

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Questions

- *What is the asymptotic distribution of non-real zeros? [Boggs, Driver, Duren, Johnston, Jordaan, Kuijlaars, Möller, Orive, Srivastava, Zhou, Wang, Martínez-Finkelshtein, Martínez-Gonzáles]*
- *When are all n zeros real and what is their location?*
- *Why are interested in real zeros?*

Klein's result

Theorem (Klein, 1890)

Let $F = {}_2F_1(-n, b; c; z)$ *where* $b,\ c \in \mathbb{R}$ *and* $c > 0$.

(i) *For* $b > c + n$, *all zeros of* F *are real and lie in* $(0, 1)$.

(ii) *For* $c + j - 1 < b < c + j$, $j = 1, 2, \dots, n$; F *has* j *real zeros in* $(0, 1)$. *If* $(n - j)$ *is odd,* F *has one additional real zero in* $(1, \infty)$.

(iii) *For* $0 < b < c$, *if* n *is odd,* F *has one real zero in* $(1, \infty)$.

(iv) *For* $-j < b < -j + 1$, $j = 1, 2, \dots, n$, F *has* j *real negative zeros. If* $(n - j)$ *is odd,* F *has one additional real zero in* $(1, \infty)$.

(v) *For* $b < -n$, *all zeros of* F *are real and lie in* $(-\infty, 0)$.

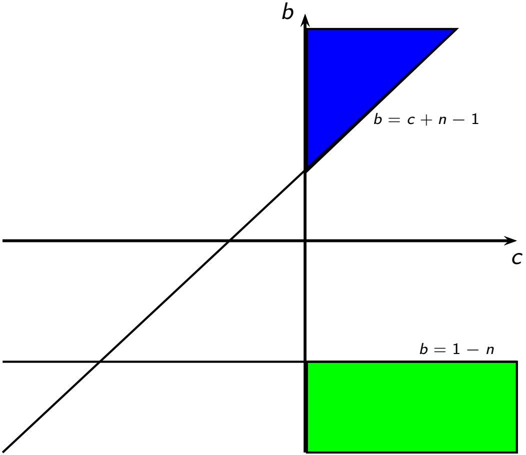

Figure: Values of b and c for which ${}_2F_1(-n, b; c; z)$ has n real simple zeros in the intervals $(0, 1)$, $(-\infty, 0)$ are indicated by the blue and green regions respectively.

Hilbert-Klein formulas

- The Hilbert-Klein formulas give number of zeros of Jacobi polynomials in the intervals $(-1, 1)$, $(-\infty, -1)$ and $(1, \infty)$.
- Stated in Szegö's book
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$$ {}_2F_1\left(\begin{array}{c} -n, \alpha+\beta+1+n \\ \alpha+1 \end{array} ; z\right) = \frac{n!}{(\alpha+1)_n} \mathcal{P}_n^{(\alpha,\beta)}(w) $$

where $w = 1-2z$:

$$ \begin{array}{rcl} 1 < w < \infty & \leftrightarrow & -\infty < z < 0 \\ -\infty < w < -1 & \leftrightarrow & 1 < z < \infty \\ -1 < w < 1 & \leftrightarrow & 0 < z < 1 \end{array} $$

Jacobi's formula

Jacobi's formula [Rodrigues, 1816; Ivory, 1822; Jacobi, 1827]

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Recall Leibnitz' formula for the n^{th} derivative of the product of two functions

$$ \frac{d^n}{dx^n}\{f(x)g(x)\} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) f^{(k)}(x) g^{n-k}(x) $$

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Theorem (Jacobi's formula)

For $n \in \mathbb{N}$

$$ {}_2F_1(-n;\ b;\ c;\ z) = \frac{z^{1-c}(1-z)^{c+n-b}}{(c)_n} \frac{d^n}{dz^n}\left[z^{c+n-1}(1-z)^{b-c}\right] \qquad (2) $$

Let $f(z) = (1 - z)^{b-c}$, then

$$
\begin{aligned}
f'(z) &= (-1)(b-c)(1-z)^{b-c-1} \\
f''(z) &= (-1)(-1)(b-c)(b-c-1)(1-z)^{b-c-2} \\
f^{(k)}(z) &= (-1)^k (b-c)(b-c-1)...(b-c-k+1)(1-z)^{b-c-k} \\
&= (c-b)(c-b+1)...(c-b+k-1)(1-z)^{b-c-k} \\
&= (c-b)_k (1-z)^{b-c-k} \qquad (3)
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Let $f(z) = (1-z)^{b-c}$, then

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Also if $g(z) = z^{c+n-1}$,

$$\begin{aligned} g'(z) &= (c+n-1)z^{c+n-2} \\ g''(z) &= (c+n-1)(c+n-2)z^{c+n-3} \\ g^{(n-k)}(z) &= (c+n-1)(c+n-2)...\left(c+n-(n-k)\right)z^{c+n-(n-k)-1} \\ &= (c+n-1)(c+n-2)...(c+k)z^{c+k-1} \\ &= \frac{(c)_n}{(c)_k}z^{c+k-1}. \qquad (4) \end{aligned}$$

Then, using Leibnitz formula, (3) and (4) the RHS of (2) is equal to

$$\begin{aligned}
&\frac{z^{1-c}(1-z)^{c+n-b}}{(c)_n}\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(z)g^{(n-k)}(z)\\
&=\frac{z^{1-c}(1-z)^{c+n-b}}{(c)_n}\sum_{k=0}^{n}\binom{n}{k}(c-b)_k(1-z)^{b-c-k}\frac{(c)_n}{(c)_k}z^{c+k-1}\\
&=\sum_{k=0}^{n}\binom{n}{k}\frac{(c-b)_k}{(c)_k}(1-z)^n\left(\frac{z}{1-z}\right)^k\\
&=(1-z)^n\sum_{k=0}^{n}\frac{(-n)_k(c-b)_k}{(c)_k k!}\left(\frac{-z}{1-z}\right)^k,\\
&=(1-z)^n\,{}_2F_1\left(-n,c-b;\;c;\;\frac{-z}{1-z}\right)\\
&=\;{}_2F_1(-n,b;\;c;\;z)\quad\text{by Pfaff's transformation}
\end{aligned}$$

Theorem (Pfaff's transformation)

If $|x|<1$ *and* $|x/(1-x)|<1$, $c>b$,

$${}_2F_1(a,b;c;x)=(1-x)^{-a}{}_2F_1\left(a,c-b;c;\frac{-x}{1-x}\right).$$

The orthogonality of the polynomial $F(z) = {}_2F_1(-n, b; c; z)$ follows in a transparent way from Rodrigues' formula and it is interesting to see how the interval of orthogonality varies with the parameters b and c.

Theorem

Let $n \in \mathbb{N}_0$, b, $c \in \mathbb{R}$ and $-c \notin \mathbb{N}_0$. Then $F(z) = {}_2F_1\left(-n, b; c; z\right)$ is the n^{th} degree orthogonal polynomial for the n-dependent positive weight function $|z^{c-1}(1-z)^{b-c-n}|$ on the intervals

(i) $(0, 1)$ *for* $c > 0$ *and* $b > c + n - 1$;

(ii) $(1, \infty)$ *for* $c + n - 1 < b < 1 - n$;

(iii) $(-\infty, 0)$ *for* $c > 0$ *and* $b < 1 - n$

and has exactly n real, simple zeros on these intervals.

Proof of (i)

We must show that if $g_\ell(z)$ is an arbitrary polynomial of degree $\ell < n$, then for $c > 0$, $b > c + n - 1$, we have

$$\int_0^1 F(z) g_\ell(z) z^{c-1} (1-z)^{b-c-n} dz = 0$$

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Now from Rodrigues' formula (2),

$$(c)_n z^{c-1}(1-z)^{b-c-n} F(z) = D^n \left[z^{c+n-1}(1-z)^{b-c} \right]$$

where $D^n = \dfrac{d^n}{dz^n}$. Thus

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$$\int_0^1 \left\{ D^n \left[z^{c-1+n}(1-z)^{b-c} \right] \right\} g_\ell(z)\, dz = (-1)^n \int_0^1 z^{c-1+n}(1-z)^{b-c} D^n \left[g_\ell(z) \right] dz$$
$$+ \sum_{k=1}^{n} (-1)^{k-1} D^{n-k} \left[z^{c-1+n} (1-z)^{b-c} \right] D^{k-1} \left[g_\ell(z) \right] \Big|_{z=0}^{z=1}$$

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Each term in the sum (5) contains a product of powers of z and powers of $(1-z)$, where the lowest and highest powers of z are c and $(c+n-1)$ respectively. The lowest and highest powers of $(1-z)$ are $(b-c-n+1)$ and $(b-c)$ respectively.

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Hence, the boundary terms will all vanish when $c > 0$ and $b > c+n-1$.

We have shown that for $c > 0$, $b > c+n-1$ and $\ell < n$,

$$
\int_0^1 F(z) g_\ell(z) z^{c-1}(1-z)^{b-c-n} dz = 0
$$

Remark

Using the hypergeometric representation of Jacobi polynomials

$$ {}_2F_1\left(\begin{matrix} -n, \alpha+\beta+1+n \\ \alpha+1 \end{matrix} ; z\right) = \frac{n!}{(\alpha+1)_n}\mathcal{P}_n^{(\alpha,\beta)}(1-2z), \tag{6} $$

we see that the orthogonality relation of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ *for* $\alpha,\ \beta > -1$ *follows on replacing*

$$ \begin{aligned} b &= \alpha+\beta+n+1 \\ c &= \alpha+1 \\ z &= \frac{1-x}{2} \end{aligned} $$

in

$$ \int_0^1 {}_2F_1(-n, b; c; z) g_\ell(z) z^{c-1}(1-z)^{b-c-n} dz = 0 $$

for $c > 0,\ b > c+n-1$.

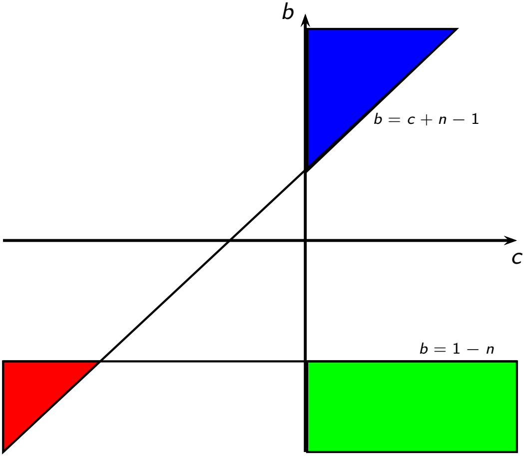

Figure: Values of b and c for which ${}_2F_1(-n, b; c; z)$ has n real simple zeros in the intervals $(0, 1)$, $(-\infty, 0)$ and $(1, \infty)$ are indicated by the blue, green and red regions.

Klein's result vs orthogonality

Klein's result:

- Pertains to general ${}_2F_1$ functions
- Proof uses complex geometric argument
- Results for other parameter values follow from Pfaff's transformation:

$$ {}_2F_1(-n, b; c; z) = \frac{(c-b)_n}{(c)_n} {}_2F_1(-n, b; 1-n+b-c; 1-z) $$

  - All zeros of F lie in $(-\infty, 0)$ for $b < -n+1$ and $c > 0$

  is equivalent to

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Klein's result and orthogonality yield the same parameter values.

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Answer: [Dominici, Johnston, & Jordaan]

- An algorithm based on a modification of the division algorithm [Schmeisser, 1993] extends parameter values where ${}_2F_1$ polynomials have only real zeros

- The location of the real zeros for these parameter values can be obtained.

The algorithm

Recall that given two polynomials $f(x)$ and $g(x)$, with $\deg(f) \geq \deg(g)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = q(x)g(x) + r(x)$$

with $\deg(r) < \deg(g)$.

Denote the leading coefficient of a polynomial $f(x) = a_n x^n + an - 1x^{n-1} + \cdots + a_0$ by $lc(f) = a_n$.

Let $f(x)$ be a real polynomial with $\deg(f) = n \geq 2$.
Define

$$f_0(x) := f(x) \qquad \text{and} \qquad f_1(x) := f'(x)$$

and proceed for $k = 1, 2, \ldots$ as follows.
If $\deg(f_k) > 0$ perform the division of f_{k-1} by f_k to obtain

$$f_{k-1}(x) = q_{k-1}(x) f_k(x) - r_k(x).$$

Define

$$f_{k+1}(x) = \begin{cases} r_k(x) & \text{if } r_k(x) \not\equiv 0 \\ f_k'(x) & \text{if } r_k(x) \equiv 0. \end{cases}$$

Terminate the algorithm when f_k is constant and generate the sequence of numbers $c_1, c_2, \ldots$ where

$$c_k = \begin{cases} \dfrac{lc(f_{k+1})}{lc(f_{k-1})} & \text{if } r_k(x) \not\equiv 0 \\ 0 & \text{if } r_k(x) \equiv 0 \end{cases}.$$

The theorem

With the same notation as for the algorithm, we have

Theorem (Rahman & Schmeisser, 2002)

Let f be a polynomial of degree n with real coefficients. Then

1. *f has only real zeros if and only if the above algorithm produced $n-1$ non-negative numbers $c_1, \ldots, c_{n-1}$.*
2. *The zeros of f are all real and simple if and only if the numbers $c_1, \ldots, c_{n-1}$ are all positive.*

Applying the algorithm to ${}_2F_1$ polynomials

Computational implementation using Maple 13 shows that the restrictions placed on the ranges of parameters b, $c \in \mathbb{R}$ given by Klein's result and orthogonality are not the best possible and that there are other values of b, $c \in \mathbb{R}$ for which ${}_2F_1\left(-n, b; c; z\right)$ have n real simple zeros.

The results obtained are proven analytically.

The intervals where the real zeros are located for the "new" values of b and c are determined.

Proposition

- *The zeros of* ${}_2F_1(-2, b; c; z)$ *are real and simple if and only if either:*

  (i) $c < -1$ *and* $c < b < 0$.
  (ii) $-1 < c < 0$ *and* $b > 0$ *or* $b < c$.
  (iii) $c > 0$ *and* $b < 0$ *or* $c < b$.

- *The zeros of* ${}_2F_1(-3, b; c; z)$ *are real and simple if and only if either:*

  (i) $c < -2$ *and* $1 + c < b < -1$.
  (ii) $-2 < c < -1$ *and* $-1 < b < 1 + c$.
  (iii) $c > -1, c \neq 0$ *and* $b < -1$ *or* $b > c + 1$.

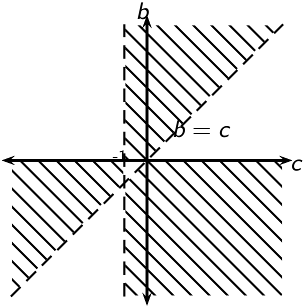

Figure: Values of b and c for which ${}_2F_1(-2, b; c; z)$ has only real simple zeros

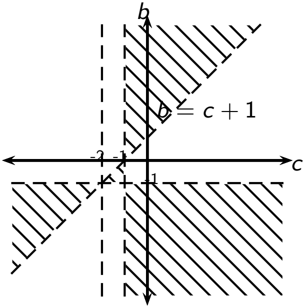

Figure: Values of b and c for which ${}_2F_1(-3, b; c; z)$ has only real simple zeros

Main result

Theorem

The zeros of ${}_2F_1(-n, b; c; z)$ *are real and simple for all* $n \geq 4$ *if and only if*

$$\begin{aligned} (c,b) &\in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4 \text{ where} \\ \mathcal{R}_1 &= \{c+n-2 < b < 2-n\}, \\ \mathcal{R}_2 &= \{c > -1, \quad b < 2-n\}, \\ \mathcal{R}_3 &= \{c > -1, \quad b > n-2, \quad b > c+n-2\}, \\ \mathcal{R}_4 &= \{-1 < c < 0, \quad c+n-2 < b < n-2\}. \end{aligned}$$

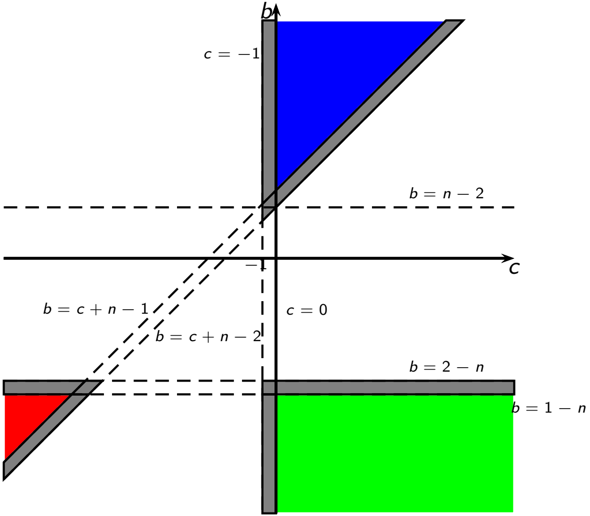

Figure: Values of b and c for which ${}_2F_1(-n, b; c; z)$, $n = 4, 5, \dots$ has n real simple zeros

Real zeros of ${}_2F_1$ polynomials

Question:

- Why are real zeros important?

Real zeros of ${}_2F_1$ polynomials

Question:

- Why are real zeros important?

Answer:

- **Applications:**
  - Canonical divisors in weighted Bergman spaces: Proof of the main result depended on knowledge of the location of the zeros of a ${}_2F_1$ function [Weir, 2002]
  - Poles and convergence of Padé approximants for ${}_2F_1(a, 1; c; z)$