# Classical Inductive Logic, Carnap’s Programme and the Objective Bayesian Approach

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# 1 Classical Inductive Logic

Consider the following argument in propositional logic:

$$\begin{array}{r} a \to b \\ b \\ \hline a \end{array}$$

We can ask whether the argument is deductively valid:

$$a \to b, b \models a?$$

We know the argument is invalid by considering its truth table:

| $a$ | $b$ | $a \to b$ | $b$ | $a$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | $\boxed{\text{T}}$ | $\boxed{\text{T}}$ | $\boxed{\text{F}}$ |
| F | F | T | F | F |

Hence this argument, **affirming the consequent**, is deemed fallacious: $a \to b, b \not\models a$.

While the argument seems a poor one from a deductive point of view, we can ask:

**Partial Entailment.** To what extent is the conclusion *plausible*, given the premisses?

IE What level $y$ of plausibility attaches to the conclusion, given the premisses?

$$a \rightarrow b, b \mathrel{|\!\approx} a^y.$$

**Support.** To what extent do one or more premisses make the conclusion *more* plausible than it is in their absence?

IE To what extent do the those premisses *support* the conclusion?

EG Compare $y$ and $z$ where $a \rightarrow b, b \mathrel{|\!\approx} a^y$ and $a \rightarrow b \mathrel{|\!\approx} a^z$.

**Classical inductive logic.** Degree of partial entailment is the proportion of those truth assignments which make the premisses true that also make the conclusion true (Wittgenstein, 1922, §5.15).

This proportion can be read off a standard truth table (Wittgenstein, 1922, §5.151):

| $a$ | $b$ | $a \to b$ | $b$ | $a$ |
|---|---|---|---|---|
| T | T | [T] | [T] | [T] |
| T | F | F | F | T |
| F | T | [T] | [T] | F |
| F | F | T | F | F |

$$a \to b, b \mid\!\approx a^{1/2}.$$

Classical inductive logic follows naturally from the classical interpretation of probability:

> The theory of chance consists in reducing all the events of the same kind to a certain number of cases equally possible, that is to say, to such as we may be equally undecided about in regard to their existence, and in determining the number of cases favorable to the event whose probability is sought. The ratio of this number to that of all the cases possible is the measure of this probability, which is thus simply a fraction whose numerator is the number of favorable cases and whose denominator is the number of all the cases possible. (Laplace, 1814, pp. 6–7.)

With one premiss, $a \rightarrow b$, we have the following probability table:

| $a$ | $b$ | $a \rightarrow b$ | $a$ |
|---|---|---|---|
| T | T | $\boxed{\text{T}}$ | $\boxed{\text{T}}$ |
| T | F | F | T |
| F | T | $\boxed{\text{T}}$ | F |
| F | F | $\boxed{\text{T}}$ | F |

Hence,

$$a \rightarrow b \mid\approx a^{1/3}.$$

∴ The degree to which $b$ supports the conclusion $a$, relative to $a \rightarrow b$, is $1/2 - 1/3 = 1/6$.

Polya (1954, §12.1) calls this the **fundamental inductive pattern**:

- 'the verification of a consequence renders a conjecture more credible.'

EG Consider the following variant:

$$\begin{array}{r} (a \wedge c) \to b \\ b \\ \hline a \wedge c \end{array}$$

This has truth table:

| $a$ | $b$ | $c$ | $(a \wedge c) \to b$ | $b$ | $a \wedge c$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | T | F |
| T | F | T | F | F | T |
| T | F | F | T | F | F |
| F | T | T | T | T | F |
| F | T | F | T | T | F |
| F | F | T | T | F | F |
| F | F | F | T | F | F |

The truth table shows that:

- This is an invalid argument (lines 2,5,6).
- $(a \wedge c) \to b, b \mid\approx a \wedge c^{1/4}$ (lines 1,2,5,6).
- $(a \wedge c) \to b \mid\approx a \wedge c^{1/7}$ (lines 1,2,4,5,6,7,8).

∴ $b$ supports $a \wedge c$: $1/4 - 1/7 = 3/28$.

# Examining a Possible Ground

Polya (1954, §13.2) puts forward the following inference pattern:

$$\frac{\begin{array}{r} a \text{ implied by } b \\ b \text{ false} \end{array}}{a \text{ less credible}}$$

| $a$ | $b$ | $b \to a$ | $\neg b$ | $a$ |
|---|---|---|---|---|
| T | T | T | F | T |
| T | F | $\boxed{\text{T}}$ | $\boxed{\text{T}}$ | $\boxed{\text{T}}$ |
| F | T | F | F | F |
| F | F | $\boxed{\text{T}}$ | $\boxed{\text{T}}$ | F |

$$b \to a, \neg b \mid\!\approx a^{1/2}.$$

$$b \to a \mid\!\approx a^{2/3}.$$

So the premiss $\neg b$ **undermines** the conclusion $a$:

$$\neg b \searrow_{25\%} a \quad [b \to a].$$

# Analogy

Polya (1954, §13.9) proposes:

$$\frac{\begin{array}{r} a \text{ analogous to } b \\ b \text{ true} \end{array}}{a \text{ more credible}}$$

Here '$a$ is analogous to $b$' is understood as there being some common ground $g$, such that $g \rightarrow a$ and $g \rightarrow b$. I.e.,

$$\frac{\begin{array}{r} g \rightarrow a \\ g \rightarrow b \\ b \text{ true} \end{array}}{a \text{ more credible}}$$

Classical inductive logic validates such an inference:

$$g \to a, g \to b, b \mid\!\approx a^{2/3}.$$

| $g$ | $a$ | $b$ | $g \to a$ | $g \to b$ | $b$ | $a$ |
|---|---|---|---|---|---|---|
| T | T | T | $\boxed{T}$ | $\boxed{T}$ | $\boxed{T}$ | $\boxed{T}$ |
| T | T | F | T | F | F | T |
| T | F | T | F | T | T | F |
| T | F | F | F | F | F | F |
| F | T | T | $\boxed{T}$ | $\boxed{T}$ | $\boxed{T}$ | $\boxed{T}$ |
| F | T | F | T | T | F | T |
| F | F | T | $\boxed{T}$ | $\boxed{T}$ | $\boxed{T}$ | F |
| F | F | F | T | T | F | F |

$$g \to a, g \to b \mid\!\approx a^{3/5}.$$

$$b \nearrow^{16.\dot{6}\%} a \quad [g \to a, g \to b].$$

CIL also validates the inference if the first two premisses are not held fixed:

$$\mid\!\approx a^{1/2},$$

$$g \to a, g \to b, b \nearrow^{33.\dot{3}\%} a.$$

# Why Inductive Logic?

## Decision Making

Bayesian decision theory determines an act from a utility matrix and relevant probabilities:

| | $r$ | $\neg r$ |
|---|---|---|
| chemotherapy | +6 | -2 |
| radiotherapy | +4 | -1 |

Decision theory says that one should perform an act with maximum expected utility.

∴ Give chemotherapy if $6P(r) - 2P(\neg r) > 4P(r) - 1P(\neg r)$.

IE If $P(r) > 1/3$.

How do we get the relevant probabilities?

- Need to determine $P(r)$ from available evidence.

  EG $m, m \wedge b \rightarrow r$
  - $m$ = metastasised, $b$ = biological marker.

∴ We need inductive logic:

$$m, m \wedge b \rightarrow r \mathrel{|\!\!\approx} r^?$$

| $m$ | $b$ | $r$ | $(m \wedge b) \rightarrow r$ | $m$ | $r$ |
|---|---|---|---|---|---|
| T | T | T | $\boxed{T}$ | $\boxed{T}$ | $\boxed{T}$ |
| T | T | F | F | T | F |
| T | F | T | $\boxed{T}$ | $\boxed{T}$ | $\boxed{T}$ |
| T | F | F | $\boxed{T}$ | $\boxed{T}$ | F |
| F | T | T | T | F | T |
| F | T | F | T | F | F |
| F | F | T | T | F | T |
| F | F | F | T | F | F |

$$m, m \wedge b \rightarrow r \mid\!\approx r^{2/3}.$$

∴ Giving chemotherapy maximises expected utility.

In general, decision theory requires probabilities and utilities in order to make a decision.

- We need inductive logic to determine the required probabilities from evidence.

## Artificial intelligence

> *Something like Carnap's theory [of inductive logic] would be required if an electronic reasoning machine is ever built.* (Good, 1950, p. 48.)

∵ The concept of partial entailment is of very wide applicability in AI.

Consider a partial entailment relation of the form $\varphi_1, \ldots, \varphi_k \approx \psi^y$.

**Medical Decision Support.** E.g., cancer treatment example above.

EG A doctor may need to decide whether to prescribe you statins.

- $\psi$ = the hypothesis that you will otherwise develop cardiovascular disease.
- $\varphi_1, \ldots, \varphi_k$ = background information about the drug and its effects, and features of you, such as your cholesterol levels and your blood pressure.

EG MYCIN, an expert system for the diagnosis and treatment of infections, was one of the earliest medical systems to incorporate numerical ‘certainty factors’.

**Robotics.** $\psi$ = the hypothesis that the robot is in a certain location.

- $\varphi_1, \ldots, \varphi_k$ = background information about its environment and observations from its sensory apparatus.

**Financial Decision Support.** A bank system may need to decide to provide whether to give you a loan.

- $\psi$ = the hypothesis that you will be able to pay the money back.
- $\varphi_1, \ldots, \varphi_k$ = general bank rules and your banking history.

**Bioinformatics.** $\psi$ = a hypothesis about genetic linkage in the fruit fly *Drosophila*.

- $\varphi_1, \ldots, \varphi_k$ = DNA data.

**Natural Language Processing.** $\psi$ = the hypothesis that a word has a certain meaning.

- $\varphi_1, \ldots, \varphi_k$ = information about the language and the preceding words.

## The Quest for the Grail

The prospect of a viable inductive logic is extremely attractive.

The search for a viable inductive logic is the quest for a **g**eneral, **r**easonable, **a**pplicable **i**nductive **l**ogic, or **grail** for short.

**General.** It should be able to generate the full variety of inductive inferences.

**Reasonable.** It should be well-motivated and should yield rational inferences.

**Applicable.** E.g., to decision making and AI.

**Inductive.** It should handle non-deductive inferences.

**Logic.** Inferences should be based on argument and sentence structure.

Classical inductive logic appears promising in certain respects:

- ✓ It yields several inferences that are reasonable in that they accord with intuition.
- ✓ The truth-table method suggests that it should be easily applicable, at least to small problems.
- ✗ Unfortunately, though, classical inductive logic is not sufficiently general, as we shall see next.

The grail is not so easily obtained.

# Learning from experience

Carnap (1945, p. 81): classical inductive logic fails to allow learning from experience.

$$\approx Br_{101}^{1/2}$$

| $Br_{101}$ |
|---|
| T |
| F |

That seems reasonable. But

$$Br_1, \ldots, Br_{100} \approx Br_{101}^{1/2}$$

since $Br_1, \ldots, Br_{100}$ are logically independent of $Br_{101}$.

| $Br_1$ | $Br_2$ | ... | $Br_{100}$ | $Br_{101}$ |
|---|---|---|---|---|
| T | T | ... | T | T |
| T | T | ... | T | F |
| T | T | ... | F | T |
| T | T | ... | F | F |
| ... | ... | ... | ... | ... |

∴ Degree of support = 0.

IE This represents an inability to learn from experience.

NB This observation had previously been made by George Boole and Wittgenstein.

An inductive logic needs to capture:

**Inductive Entailment.** The degree to which an observed sample of ravens makes plausible the proposition that the next raven is black.

- An ampliative concept that can link logically independent propositions.
- CIL fails to capture this concept.
- Carnap (1945) abandoned the classical notion of partial entailment to try to capture learning from experience.

**Logical Entailment.** The degree to which, e.g., $A \vee B$ makes plausible proposition $A$.

- A non-ampliative concept, attributable to logical dependence.
- CIL is required to capture this concept (Wittgenstein, Kemeny and Oppenheim).
- Kemeny and Oppenheim (1952) focused on classical partial entailment to the exclusion of learning from experience.

Salmon argued that it is not possible to capture both phenomena in a single inductive logic:

> if degree of confirmation is to be identified with partial [logical] entailment, then $c^{\dagger}$ [i.e., CIL] is the proper confirmation function after all, for it yields the result that *p* is probabilistically irrelevant to *q* whenever *p* and *q* are completely independent and there is no partial [logical] entailment between them. . . . Unfortunately for induction, statements strictly about the future (unobserved) are completely independent of statements strictly about the past (observed). Not only are they deductively independent of each other, but also they fail to exhibit any partial [logical] entailment. The force of Hume's insight that the future is logically independent of the past is very great indeed. It rules out both full entailment and partial [logical] entailment. If [logical] entailment were the fundamental concept of inductive logic, then it would in fact be impossible to learn from experience. (Salmon, 1967, pp. 731–2.)

# 2 Carnap's Programme

## Conditionalising on a Blank Slate

- Premisses $\varphi_1, \ldots, \varphi_k$ are categorical sentences of a monadic predicate language $\mathcal{L}$.
  - Let $\varphi \stackrel{\text{df}}{=} \varphi_1 \wedge \cdots \wedge \varphi_k$.
  - The **atoms** $\alpha(x)$ have the form $\pm U_1(x) \wedge \cdots \wedge \pm U_m(x)$.
  - The ***n*-states** $\Omega_n = \{\alpha_1(t_1) \wedge \cdots \wedge \alpha_n(t_n)\}$.

**Probabilism.** $\varphi_1, \ldots, \varphi_k \approx\!\!\!\!| \; \psi^y$ if and only if $P_\varphi(\psi) = y$ for some suitable probability function $P_\varphi$ which best fits the premisses.

**Conditionalisation.** Identify $P_\varphi(\psi) = P_\varnothing(\psi|\varphi)$.

**Blank slate.** Find an appropriate $P_\varnothing$ that corresponds to the situation in which there is no information available.

The obvious choice of blank slate is the **equivocator** on $\mathcal{L}$:

$$P_\varnothing(\omega) = P_=(\omega) \stackrel{\text{df}}{=} \frac{1}{|\Omega_n|} \text{ for all } \omega \in \Omega_n.$$

$\therefore$ $\varphi_1, \ldots, \varphi_k \approx\!\!\!\!| \; \psi^y$ if and only if $P_=(\psi|\varphi) = y$.

NB This is essentially classical inductive logic.

Carnap tried to capture inductive entailment by finding different blank slate functions.

- This led him to the Johnson-Carnap continuum of inductive methods:

IE $P_{\varnothing}(\psi) = c_\lambda(\psi)$, where $c_\lambda$ is defined by:

$$c_\lambda(\alpha_{l+1}(t_{l+1})|\alpha_1(t_1) \wedge \cdots \wedge \alpha_l(t_l)) \ = \ \frac{\#\alpha_{l+1} + \lambda/2^m}{l+\lambda}.$$

- $\#\alpha_{l+1}$ is the number of occurrences of $\alpha_{l+1}$ in $\alpha_1, \ldots, \alpha_l$.

So

$$\varphi_1, \ldots, \varphi_k \mathrel{|\!\!\approx}_\lambda \psi^Y \text{ iff } c_\lambda(\psi|\varphi) \in Y.$$

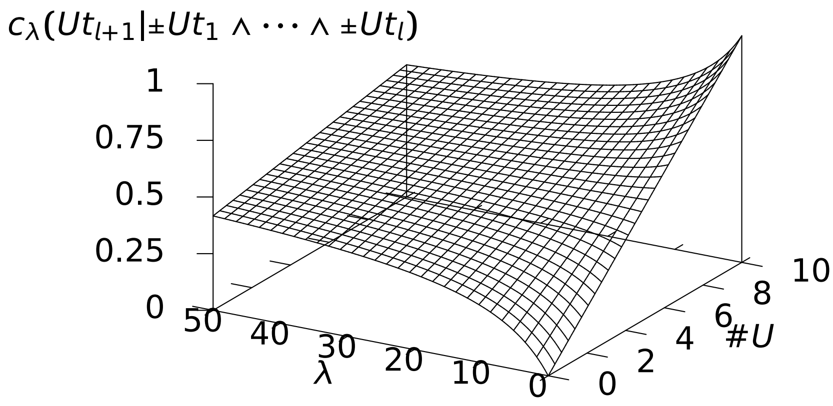


Figure 1: Carnap's inductive methods for $\lambda \in [0, 50]$, $m = 1$ and $l = 10$.

If $\mathcal{L}$ has more than one unary predicate then the $c_\lambda$ can be characterised as the only probability functions satisfying the following conditions (see, e.g., Paris, 1994, pp. 189–197):

**Permutation.** $P$ is invariant under permutations of the constant symbols $t_i$: $P(\theta(t_{i_1}, \ldots, t_{i_s})) = P(\theta(t_{j_1}, \ldots, t_{j_s}))$ for any open formula $\theta(x_1, \ldots, x_s)$.

**Sufficientness.** $P(\alpha_{l+1}(t_{l+1}) | \alpha_1(t_1) \wedge \cdots \wedge \alpha_l(t_l))$ depends only on $l$ and the number $\#\alpha_{l+1}$ of occurrences of $\alpha_{l+1}$ in $\alpha_1, \ldots, \alpha_l$.

## 2.1 Difficulties for Carnap’s Programme

- ✗ **In practice we never have zero information.**
  - ✗ We can think of an uninterpreted language (*pure* inductive logic).

> In *applied IL*, we give an interpretation of the language. We say generally that the individuals are, for example, the inhabitants of a certain town or the throws of a certain die, or the states of the weather in Los Angeles at noon on the days of one year. *. . .*
>
> In contrast, in *pure IL*, we describe a language system in an abstract way, without giving an interpretation of the nonlogical constants (individual and predicate constants). (Carnap, 1971, pp. 69–70.)

- ✗ **Some of the information we have is hard or impossible to make explicit as a set of propositions that can be conditionalised upon.**
  - EG Information to do with the interpretation of the language.
  - EG Contextual or background information such as ${U_1 t_3}^{0.9}$.
  - ✗ One can enrich the language.
    - ✗ That may make the logic much more complicated and inference less tractable.
    - ✗ Problems can remain if the information imposes multiple constraints:
      - EG $U_1 t_3^{0.9}$ imposes the constraints $P(U_1 t_3) = 0.9$ and $P({U_1 t_3}^{0.9}) = 1$.
      - ∴ One would need $P(U_1 t_3 | {U_1 t_3}^{0.9}) = 0.9$.
      - • This is a further substantive condition, analogous to the Principal Principle.

✗ **Conditionalisation requires that all evidence have non-zero probability.**

- But, e.g., most plausibly $P_{\emptyset}(U_1 t_3{}^{0.9}) = 0$.
  - ✗ One could apply Jeffrey conditionalisation or minimum-cross entropy updating.
    - ✗ This won't handle more complex statements like $U_1 t_3{}^{.9} \vee U_1 t_2{}^{[.7,.8]}$.
    - ✗ This does not help in the case of categorical statements with zero probability.
    - EG Plausibly, universal generalisations.
- The premisses may be inconsistent with background information.

✗ **The permutation postulate / exchangeability is only appropriate in some circumstances.**

- Popper (1983, pp. 303–305); Good (1965, pp. 13–14) and Gillies (2000, pp. 77–83): exchangeable probabilities are only appropriate when the events are objectively independent:

EG A fair coin is tossed and 700 heads and 2 tails are observed. $c_\lambda(Ht_{703} \mid Ht_1 \wedge \cdots \wedge Ht_{700} \wedge \neg Ht_{701} \wedge \neg Ht_{702}) = \frac{700+\lambda/2}{702+\lambda}$. If $\lambda = 2$, i.e., Laplace's rule of succession, then we have that $c_\lambda(Ht_{703} \mid Ht_1 \wedge \cdots \wedge Ht_{700} \wedge \neg Ht_{701} \wedge \neg Ht_{702}) = 701/704 \approx 0.996$, which seems quite reasonable.

- Consider a case of dependence: the game of red or blue. A fair coin is tossed, changing a score $s$ to $s+1$ if heads or $s-1$ if tails. If $s \geq 0$ the result of the toss is blue, if $s < 0$ the result is red.

- Note that while the tosses of the coin are independent, the outcomes red and blue are highly dependent.

- Suppose we get a sequence of 700 blues then two reds.

- We can deduce that now $s = -2$ so objectively $P(Bt_{703} \mid Bt_1 \wedge \cdots \wedge Bt_{700} \wedge \neg Bt_{701} \wedge \neg Bt_{702}) = 0$.

- But if probabilities satisfy exchangeability then one will get a positive probability for blue.

  EG Applying Laplace's rule of succession, $P(Bt_{703} \mid Bt_1 \wedge \cdots \wedge Bt_{700} \wedge \neg Bt_{701} \wedge \neg Bt_{702}) = 701/704 \approx 0.996$.

✗ **No unproblematic way to choose the parameter $\lambda$.**

- Carnap (1952, §18): $\lambda$ will depend on empirical performance, simplicity and formal elegance.
  - ✗ No clear indication as to how to balance these considerations.
  - ✗ Blank slate: there is no evidence of empirical performance.
    - $c_0$ and $c_\infty$ are surely the simplest and most elegant.
    - Arguably one should reject $c_0$ because is leads to absurd commitments.
    - But then we are left with $c_\infty$ and classical inductive logic.
- Good (1980): treat parameters like $\lambda$ as meta-inductive parameters—attach a probability distribution and update.
  - **?** What are these probabilities of?
    - **?** That $\lambda$ is the 'true' value of the parameter? Hard to make sense of such a claim.
    - **?** That $\lambda$ offers the best balance between empirical performance, simplicity and formal elegance? It is hard to find objective standards here.
  - ✗ Prone to a regress problem.
- Carnap (1952, §§19–24) and Kuipers (1986): take an arbitrary initial value of $\lambda$ and change that as evidence $E$ is gathered in order to minimise the distance between the probability function $P_E$ and the frequency function $P^*$.
  - ✓ No regress problem.
  - ✗ $\lambda$ varies so we get a confirmation function not in the continuum.
    - ∴ Inadequate on Carnap's own account.

## 2.2 The Principle of Indifference

The Permutation Postulate can be thought of as an application of the Principle of Indifference:

> The principle of indifference asserts that if there is no *known* reason for predicating of our subject one rather than another of several alternatives, then relatively to such knowledge the assertions of each of these alternatives have an *equal* probability. (Keynes, 1921, p. 45.)

- In particular, all sequences of $l$ outcomes with the same number of positive outcomes have the same probability.

For example,

$$P_{\emptyset}(Ut_1 \wedge Ut_2 \wedge \neg Ut_3) = P_{\emptyset}(Ut_1 \wedge \neg Ut_2 \wedge Ut_3) = P_{\emptyset}(\neg Ut_1 \wedge Ut_2 \wedge Ut_3)$$

$$P_{\emptyset}(Ut_1 \wedge \neg Ut_2 \wedge \neg Ut_3) = P_{\emptyset}(\neg Ut_1 \wedge Ut_2 \wedge \neg Ut_3) = P_{\emptyset}(\neg Ut_1 \wedge \neg Ut_2 \wedge Ut_3)$$

POI has been widely criticised as giving inconsistent recommendations when applied to different partitions.

EG {red, not red} vs {red, blue, green, yellow}.

Keynes resolved this problem by insisting that we need to apply POI to the *finest* partition of alternatives:

> it is a necessary condition for the application of the principle, that these should be, relatively to the evidence, *indivisible* alternatives...
>
> The principle of indifference is not applicable to a pair of alternatives, if we know that either of them is capable of being further split up into a pair of possible but incompatible alternatives of the same form as the original pair. (Keynes, 1921, pp. 65–66.)

- In propositional logic, the finest partition is the partition of states corresponding to lines in a truth table.
- In predicate logic, there is a finest partition of $n$-states for each $n$.

So to apply the Principle of Indifference coherently, we should have:

**State Exchangeability.** For any $n$, $P_\emptyset$ should give the same probability to each $n$-state $\omega \in \Omega_n$.

$$\begin{aligned} P_\emptyset(Ut_1 \wedge Ut_2 \wedge \neg Ut_3) &= P_\emptyset(Ut_1 \wedge \neg Ut_2 \wedge Ut_3) \\ &= P_\emptyset(\neg Ut_1 \wedge Ut_2 \wedge Ut_3) \\ &= P_\emptyset(Ut_1 \wedge \neg Ut_2 \wedge \neg Ut_3) \\ &= P_\emptyset(\neg Ut_1 \wedge Ut_2 \wedge \neg Ut_3) \\ &= P_\emptyset(\neg Ut_1 \wedge \neg Ut_2 \wedge Ut_3) \\ &= P_\emptyset(Ut_1 \wedge Ut_2 \wedge Ut_3) \\ &= P_\emptyset(\neg Ut_1 \wedge \neg Ut_2 \wedge \neg Ut_3) \end{aligned}$$

$\therefore$ $P_\emptyset(\omega) = P_=(\omega) = 1/|\Omega_n|$ for all $\omega \in \Omega_n$.

Thus this consideration motivates $P_=$ and Classical Inductive Logic.

NB The only member of Carnap's continuum that satisfies State Exchangeability is $c_\infty$.

Note that Keynes' resolution restricts the application of POI to discrete partitions.

Paris (2015) advocates an alternative resolution to problems arising from the application of POI to continuous partitions.

- This involves restricting POI to those symmetries that can be represented by automorphisms of the language.

✓ This restriction of POI still yields the Permutation Postulate.

✓ It this avoids certain paradoxes on continuous domains.

✗ It does not help Carnap's programme, because it isolates $c_0$ as uniquely rational (Paris and Vencovská, 2011).

  - This function gives probability 0 to the possibility that any of a sequence of outcomes will differ from the first outcome.

    EG $c_0(L(t_2) \wedge L(t_3) \wedge \neg L(t_4) | L(t_1)) = 0$, where vehicles are observed at a road junction to see whether or not they turn left ($L$).

  - Paris and Vencovská (2011) and Paris and Vencovská (2015, Chapter 23) rightly regard this conclusion as a reductio of the automorphism approach on unary languages.

## 2.3 Which Continuum of Inductive Methods?

Where there are at least two relation symbols, the **Nix-Paris $\delta$-continuum** is the only set of probability functions satisfying the following conditions (Nix, 2005; Nix and Paris, 2006):

**Permutation.** $P$ is invariant under permutations of the constant symbols $t_i$: $P(\theta(t_{i_1}, \ldots, t_{i_s})) = P(\theta(t_{j_1}, \ldots, t_{j_s}))$ for any open formula $\theta(x_1, \ldots, x_s)$.

**Regularity.** For quantifier-free $\theta$, $P(\theta) = 0$ iff $\models \neg\theta$.

**Generalised Principle of Instantial Relevance.** If $\theta(x) \models \varphi(x)$ and $\varphi(t_{l+1}) \wedge \psi(t_1, \ldots, t_l)$ is consistent then an extra instance of $\varphi$ should not undermine $\theta(t_{l+2})$, $P(\theta(t_{l+2})|\varphi(t_{l+1}) \wedge \psi(t_1, \ldots, t_l)) \geq P(\theta(t_{l+2})|\psi(t_1, \ldots, t_l))$.

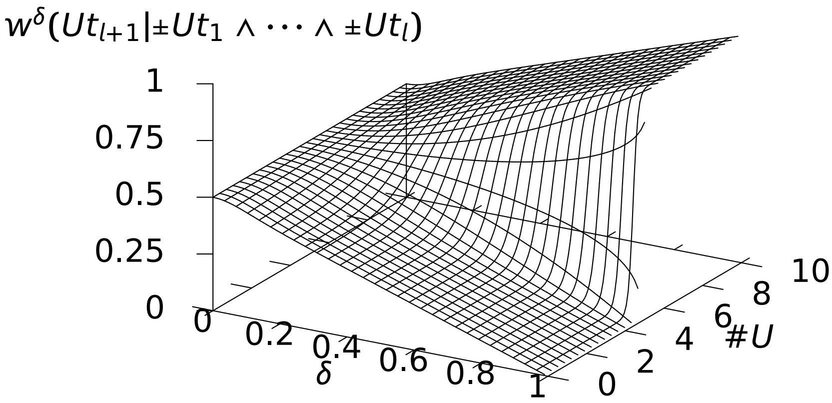


Figure 2: The Nix-Paris inductive methods for $\delta \in [0,1)$, a single unary predicate and $l = 10$.

**?** Which continuum should we endorse?

- The desiderata seem equally plausible.

∴ If we advocate any of them, we should advocate all of them.

But the equivocator $P_=$ is the only function satisfying:

**Permutation.** $P$ is invariant under permutations of the constant symbols $t_i$: $P(\theta(t_{i_1}, \ldots, t_{i_s})) = P(\theta(t_{j_1}, \ldots, t_{j_s}))$ for any open formula $\theta(x_1, \ldots, x_s)$.

**Sufficientness.** $P(\alpha_{l+1}(t_{l+1})|\alpha_1(t_1) \wedge \cdots \wedge \alpha_l(t_l))$ depends only on $l$ and the number $\#\alpha_{l+1}$ of occurrences of $\alpha_{l+1}$ in $\alpha_1, \ldots, \alpha_l$.

**Regularity.** For quantifier-free $\theta$, $P(\theta) = 0$ iff $\models \neg\theta$.

**Generalised Principle of Instantial Relevance.** If $\theta(x) \models \varphi(x)$ and $\varphi(t_{l+1}) \wedge \psi(t_1, \ldots, t_l)$ is consistent then an extra instance of $\varphi$ should not undermine $\theta(t_{l+2})$, $P(\theta(t_{l+2})|\varphi(t_{l+1}) \wedge \psi(t_1, \ldots, t_l)) \geq P(\theta(t_{l+2})|\psi(t_1, \ldots, t_l))$.

This leaves us with $P_=$ (aka $c^\dagger, c_\infty, w^0$)—i.e., Classical inductive logic.

## 2.4 Capturing Logical Entailment

Recall Salmon:

> if degree of confirmation is to be identified with [logical] entailment, then $c^\dagger$ [i.e., classical inductive logic] is the proper confirmation function after all, for it yields the result that $p$ is probabilistically irrelevant to $q$ whenever $p$ and $q$ are completely independent and there is no [logical] entailment between them. (Salmon, 1967, p. 731.)

Arguably, if $P$ is to capture logical entailment then it should render logically disjoint propositions probabilistically independent:

**Weak Irrelevance.** If quantifier-free $\theta, \varphi$ have no relation or constant symbols in common then $P(\theta|\varphi) = P(\theta)$.

But $c_0$ and $c_\infty$ are the only members of the Johnson-Carnap continuum that satisfy Weak Irrelevance (Paris and Vencovská, 2015, Chapter 20).

NB $c_0$ leads to absurd commitments.

So Salmon was right: in the framework of Carnap's programme, an inductive logic can't capture both logical entailment and inductive entailment.

# 3 Objective Bayesian Inductive Logic

## Objective Bayesian Epistemology

An agent should apportion the strengths of her beliefs according to three norms:

**Probability.** Her belief function $P_E$ should be a probability function, $P_E \in \mathbb{P}$.

**Calibration.** Her belief function should be compatible with her evidence, $P_E \in \mathbb{E} \subseteq \mathbb{P}$.

- $P_E \in \mathbb{E} = \langle \mathbb{P}^* \rangle$.

**Equivocation.** Her belief function should equivocate between basic possibilities.

- $P_E \in \text{maxent}\,\mathbb{E} = \{P \in \mathbb{E} : \text{entropy } H(P) \stackrel{\text{df}}{=} \sum_\omega P(\omega) \log P(\omega) \text{ is maximal}\}$.

NB No updating rule required: if $E$ changes to $E'$ then $P_E$ changes to $P_{E'}$.

✓ Controls worst-case expected loss (Landes and Williamson, 2013).

(Williamson, J. (2010). *In defence of objective Bayesianism*. Oxford University Press, Oxford.)

# Objective Bayesian inductive logic

$$\varphi_1^{X_1}, \ldots, \varphi_k^{X_k} \mathrel{|\!\!\!\approx^{\circ}} \psi^Y$$

iff

$$P(\psi) \in Y,$$

for any $P$ satisfying the norms of OBE,

IE For any $P \in \text{maxent}\langle\{P^* : P^*(\varphi_1) \in X_1, \ldots, P^*(\varphi_k) \in X_k\}\rangle$.

## 3.1 Capturing Logical Entailment

Objective Bayesian inductive logic preserves classical inductive logic:

**Probability.** Measure inductive plausibility by probability.

We can augment each line of a truth table with the probability that the atomic propositions take the truth values specified on that line:

| $P$ | $a$ | $b$ | $a \rightarrow b$ | $b$ | $a$ |
|---|---|---|---|---|---|
| $x_1$ | T | T | T | T | T |
| $x_2$ | T | F | F | F | T |
| $x_3$ | F | T | T | T | F |
| $x_4$ | F | F | T | F | F |

**Calibration.** These probabilities should fit the premisses.

EG States where one or more premisses turn out false should have zero probability.

| $P$ | $a$ | $b$ | $a \rightarrow b$ | $b$ | $a$ |
|---|---|---|---|---|---|
| $x_1$ | T | T | T | T | T |
| 0 | T | F | F | F | T |
| $x_3$ | F | T | T | T | F |
| 0 | F | F | T | F | F |

**Equivocation.** If the premisses fail to distinguish between two possible truth assignments, then they are equally plausible.

| $P$ | $a$ | $b$ | $a \rightarrow b$ | $b$ | $a$ |
|---|---|---|---|---|---|
| $\frac{1}{2}$ | T | T | T | T | T |
| 0 | T | F | F | F | T |
| $\frac{1}{2}$ | F | T | T | T | F |
| 0 | F | F | T | F | F |

We find then that probability $\frac{1}{2}$ attaches to the conclusion:

| $P$ | $a$ | $b$ | $a \rightarrow b$ | $b$ | $a$ |
|---|---|---|---|---|---|
| $\boxed{\frac{1}{2}}$ | T | T | T | T | $\boxed{\text{T}}$ |
| $\boxed{0}$ | T | F | F | F | $\boxed{\text{T}}$ |
| $\frac{1}{2}$ | F | T | T | T | F |
| 0 | F | F | T | F | F |

So,

$$a \rightarrow b, b \mathrel{|\!\!\approx} a^{1/2}.$$

$\therefore$ **OBIL captures logical entailment.**

## 3.2 Capturing Inductive Entailment

EG Observing ravens $r_1, ..., r_{101}$ to see if they are black $B$.

- $P_\emptyset(Br_{101}) = 1/2 = P_\emptyset(Br_{101} \mid Br_1 \wedge \cdots \wedge Br_{100})$

NB We need $P_E(Br_{101})$, where $Br_1, ..., Br_{100} \in E$.

- Suppose the agent grants that $Br_1, ..., Br_{100}$ and that outcomes are iid.
- Suppose .99 is the minimum degree to which she would need to believe $P_R^*(B) \geq x$ for her to grant it.
- Frequentist statistics can determine $\delta$ such that $P_S^*(|f_S - P_R^*(B)| \leq \delta) = .99$.
- Calibration: $P_E(1 - P_R^*(B) \leq \delta) = .99$.
- The agent grants that $P_R^*(B) \geq 1 - \delta$.
- Calibration: $P_{E'}(Br_{101}) \geq 1 - \delta$.
- Equivocation: $P_{E'}(Br_{101}) = 1 - \delta$.

$\therefore$ $\Gamma, Br_1 \wedge \cdots \wedge Br_{100} \mathrel{|\!\approx} Br_{101}^{1-\delta}$.

IE Learning from experience is possible.

NB Statistical theory is playing a crucial role here.

- Inductive entailment is captured by statistical theory and Calibration.
- Logical entailment is captured by Equivocation.

## Summary

- Partial entailment is an overloaded relation:
  - Logical entailment is captured by classical inductive logic.
  - Inductive entailment can't be captured by CIL.
- Carnap tried to capture inductive entailment by relaxing the blank slate.
  - ✗ All roads lead to $P_=$ as the natural blank slate—i.e., CIL.
- Statistical theory is best equipped to capture inductive entailment.
  - Objective Bayesian inductive logic employs statistical theory.
  - OBIL captures logical entailment by preserving CIL.

## Links

**Project.** From objective Bayesian epistemology to inductive logic, AHRC 2012–15.

See http://blogs.kent.ac.uk/jonw

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