Introduction to Lower Previsions I

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Progic Spring School
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A (mathematically) simple way to a lower prevision

$$(\Omega, \mathcal{A}, p)$$

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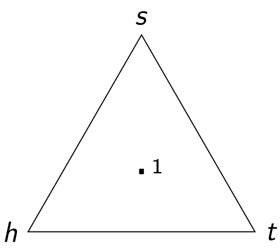
Williams (1975), Levi (1980), Walley (1991), Joyce (2008), Haenni, Romeijn, Wheeler, Williamson (2011).


Campbell's
CONDENSED
TOMATO
SOUP

Tin Can Toss

Let X be a random variable with possible values $\mathcal{X} = \{h, s, t\}$.



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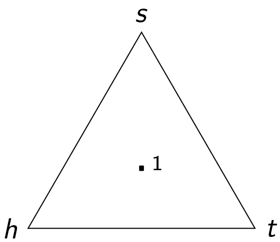

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- Example: the center point is the uniform distribution, $p_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

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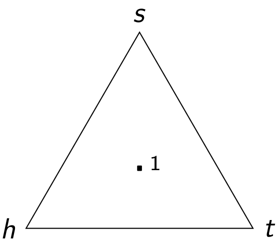

- The probability simplex includes all the probability mass functions.
- Example: the center point is the uniform distribution, $p_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
- An easy way to think about an imprecise probability model is as a **closed convex set** $\mathbb{P}$ of probability mass functions...

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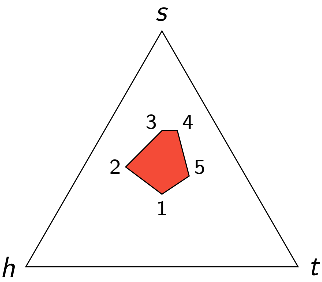

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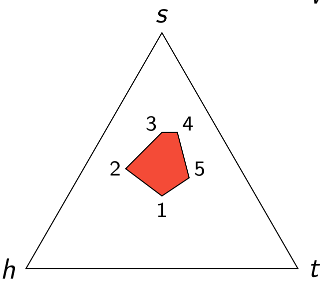

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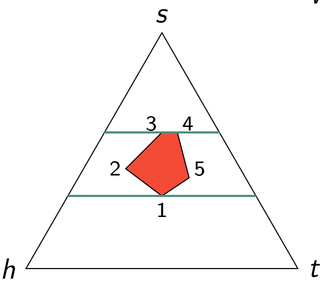

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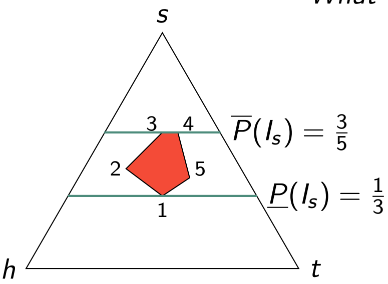

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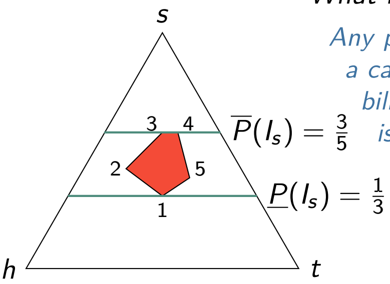

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*Any point in the red **credal set** is a candidate model for the probability of s. But, in this set there is a **lower** possible value and an **upper** possible value. The same holds for the probabilities of h and t.*

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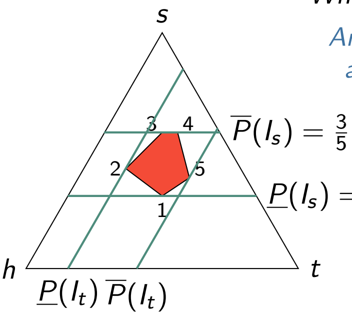

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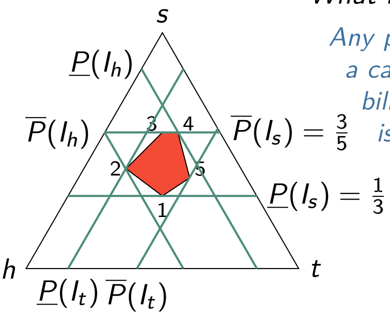

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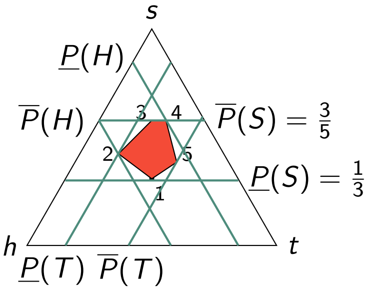

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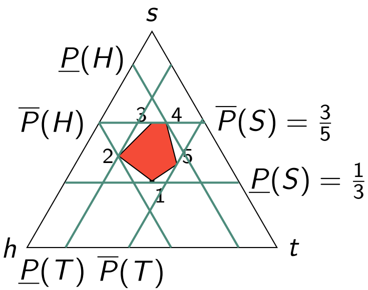

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Lower prevision (expectation)

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Conjugacy

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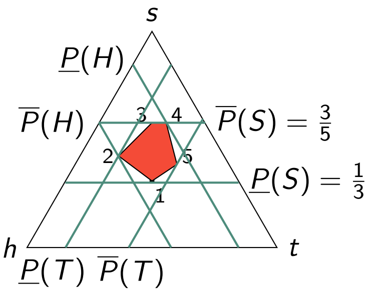

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Set of probabilities: $\mathbb{P}$

Convex closure of $\mathbb{P}$: $co(\mathbb{P})$

Lower envelope of the convex hull of $co(\mathbb{P})$: $\underline{P}(\cdot)$

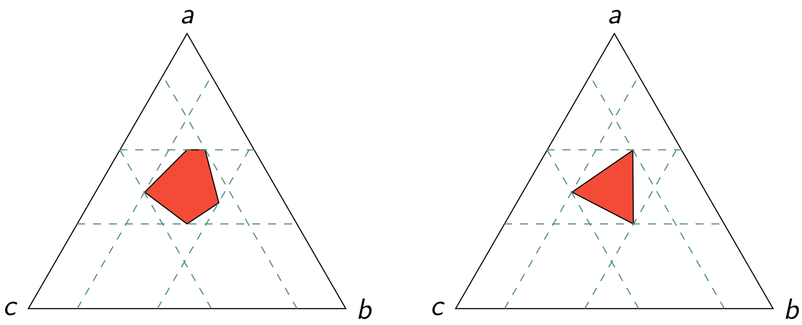
Lower Envelope and Lower Previsions

Theorem (Walley 1991)

A real functional $\underline{P}$ is a lower prevision if and only if it is the lower envelope of some credal set $\mathbb{P}$.

Many credal sets to one lower envelope

Two different credal sets, in red, that have the same lower and upper probabilities to all events.



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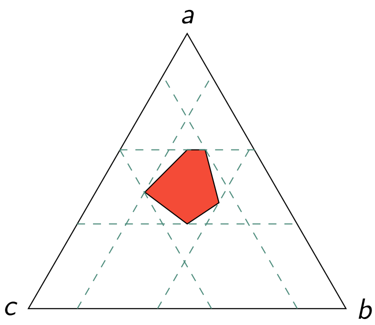

Moral: The language of events (propositions) is not expressive enough for the theory of lower previsions.

Issues with “imprecise” probabilities

- *What do these probabilities represent?*
  - Incomplete evidence?
  - Knightian uncertainty (vs risk)?
  - A person’s “credal committee”?
  - A group of Bayes agents?
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- *Is IP too problematic?*
  - Dilation & discounting cost-free information
  - No IP strictly proper scoring rules
  - Conditioning on sets of zero probabilities?
  - ...

Outline for this tutorial

Three goals

1. *Get rid of probabilities, credal sets, and lower previsions*
   - Sets of **Acceptable/Desirable Gambles**
     Williams 1977; Walley 1991, 2000; De Cooman and Troffaes 2014

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2. *Propose a benchmark for evaluating a model of belief*
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3. *Address with some alleged problems*
   - **Dilation**
   - **Violations of Good's Principle**
   - **No Strictly Proper IP Scoring Rules**

What is basic?

A set of outcomes $\mathcal{X}$.

A set of (bounded) gambles.

Bounded Gambles

A **bounded gamble** on the set $\mathcal{X}$ is a bounded real-valued map $f : \mathcal{X} \mapsto \mathbb{R}$.

- Interpreted as a **gain** $(+/-)$ that is a function of $x \in \mathcal{X}$
- The gain is expressed in units of a **linear utility** scale.
- A bounded gamble is a real-valued **random variable**.

Example: Acceptable Bounded Gamble

You **accept** f when You agree to the following transaction:

- The value x of X is determined, and
- You receive the amount $f(x)$.

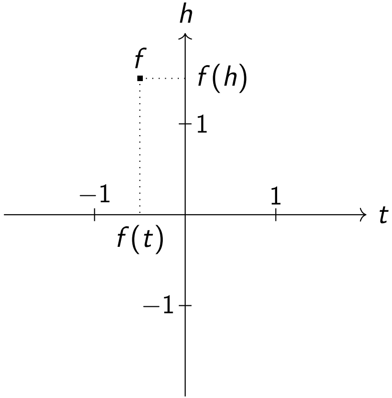
Note:

- When $f(x) \geq 0$, Your total utility will **increase** by $|f(x)|$;
- When $f(x) < 0$, Your total utility will **decrease** by $|f(x)|$.

Example: Acceptable Bounded Gamble

A **gamble** $f : \mathcal{X} \to \mathbb{R}$ is an uncertain reward whose value is $f(X)$.

Suppose X is a coin toss, $\mathcal{X} = \{h, t\}$, and You find f an **acceptable gamble**: You receive €1.50 if h and −€0.50 if t.



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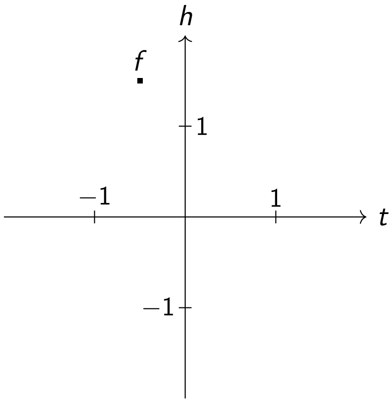

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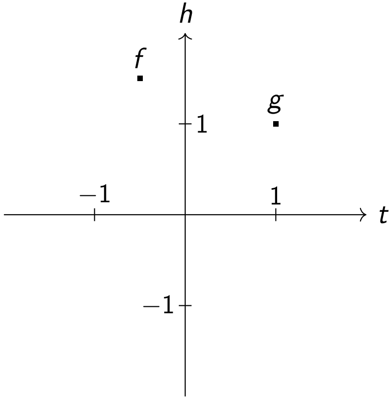

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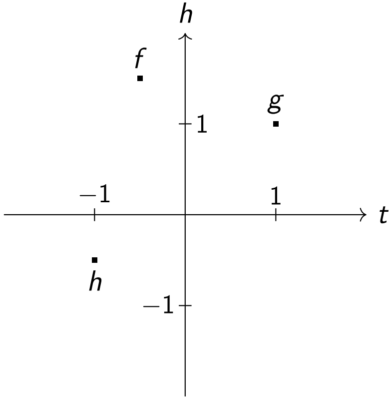

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Elicitation: So, tell me some number of gambles that You accept.

- The gambles that You accept appear as points in an n-dimensional space.
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Set of Acceptable Gambles:

$\mathcal{D} \subseteq \mathbb{B}$ is a set of gambles whose reward is no worse than zero.

Sets of Coherent Acceptable Bounded Gambles

A set of acceptable gambles $\mathcal{D}$ is **coherent** if

A1. **Avoid Partial Loss** - If $f < 0$, then $f \notin \mathcal{D}$.
Never accept when you cannot win

A2. **Accept Partial Gain** - If $f \geq 0$, then $f \in \mathcal{D}$.
Always accept when you cannot lose

A3. **Scale Invariance** - If $f \in \mathcal{D}$, then $\lambda f \in \mathcal{D}$ $(\forall \lambda \in \mathbb{R}. \lambda > 0)$
If a gamble is acceptable, a fraction of it is acceptable too

A4. **Combination** - If $f \in \mathcal{D}$ and $g \in \mathcal{D}$, then $f + g \in \mathcal{D}$.
If each is acceptable alone, they are acceptable together

Coherence for Sets of Acceptable Gambles

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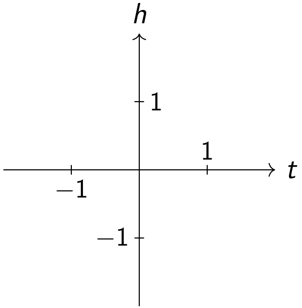

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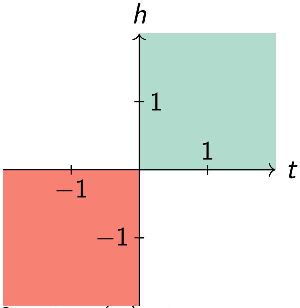

Smallest (in)coherent sets

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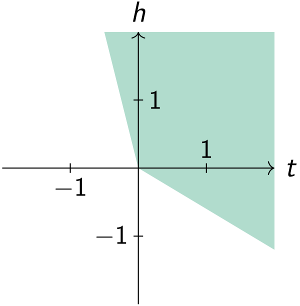

Imprecise probability model

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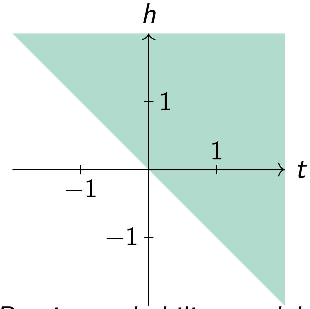

Precise probability model

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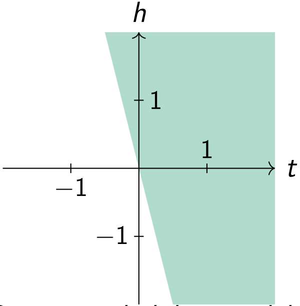

Precise probability model

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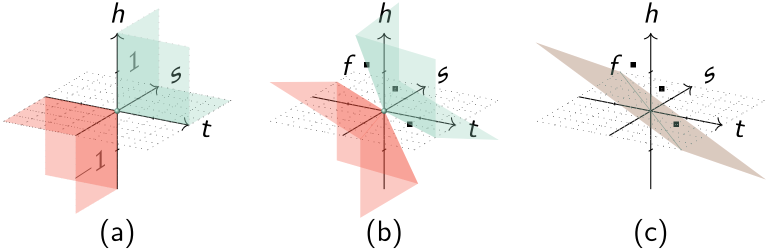
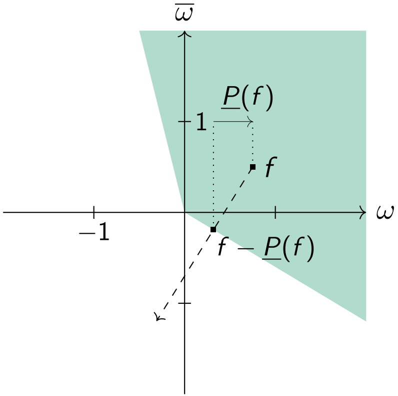


Figure: Sets of Acceptable Gambles for outcomes *heads*, *side*, and *tails*

Connecting Acceptable Gambles to Lower Previsions

$\underline{P}(f) = \sup\{\alpha \in \mathbb{R} : f - \alpha \in \mathcal{D}\}$
supremum buying price for f.

- You get uncertain f
- You give a constant α
- The result of the transaction $f - \alpha$ is acceptable to You
- The *highest price* α whereby $f - \alpha$ remains acceptable to You is Your *supremum buying price* for f.
- This is your **lower prevision** for f.



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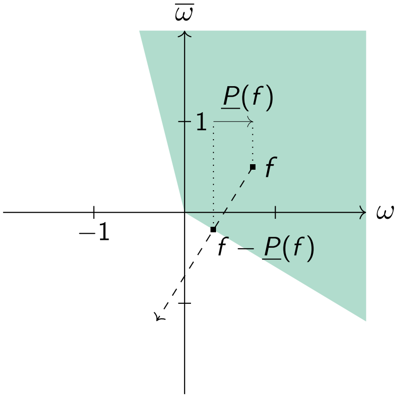
- Sets of desirable gambles are more informative than lower previsions



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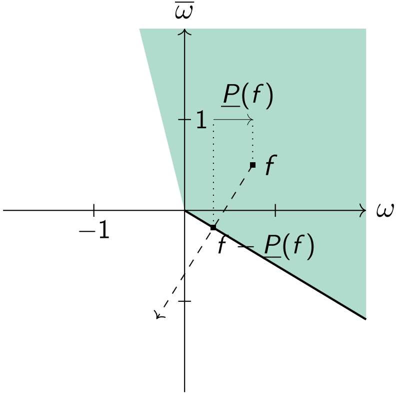
- Sets of desirable gambles are more informative than lower previsions
- The border could be **included** or



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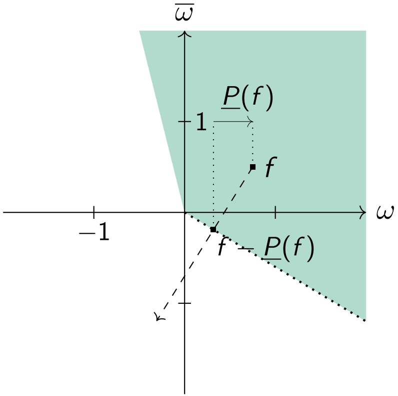

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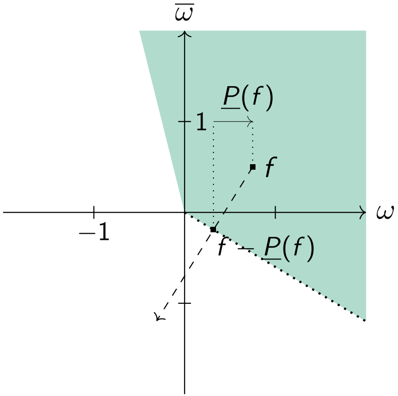
- This border behavior does not affect the supremum
- So, there are no problems with conditioning on sets of probability zero, as this occurs precisely on the border condition.



Three goals

1. *Get rid of probabilities, credal sets, and lower previsions*
   - ✓ Sets of **Acceptable Gambles**
     Williams 1977; Walley 1991, 2000; De Cooman and Troffaes 2014
2. *Propose a benchmark for evaluating a model of belief*
   - ✓ **Operationalizable**
   - ○ **Inference**: closure, marginalization, and conditionalization
   - ✓ **Unification**: unify different uncertainty models
3. *Address with some alleged problems*
   - ○ **Dilation**
   - ○ **Violations of Good's Principle**
   - ○ **No Strictly Proper IP Scoring Rules**

Inference

Probabilistic inference involves three basic components:

- Closure or "natural extension" (Walley)
- Marginalization
- Conditioning

Closure: The Natural Extension

A set of acceptable gambles $\mathcal{D}$ is **coherent**

A1. **Avoid Partial Loss** - If $f < 0$, then $f \notin \mathcal{D}$.

A2. **Accept Partial Gain** - If $f \geq 0$, then $f \in \mathcal{D}$.

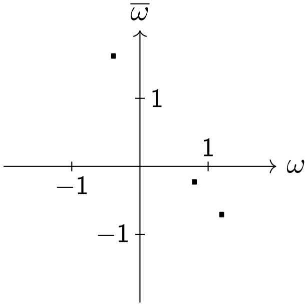
A3. **Scale Invariance** - If $f \in \mathcal{D}$, then $\lambda f \in \mathcal{D}$ $(\forall \lambda \in \mathbb{R}. \lambda > 0)$

A4. **Combination** - If $f \in \mathcal{D}$ and $g \in \mathcal{D}$, then $f + g \in \mathcal{D}$.

These axioms are viewed as production rules.

Inference through natural extension

Suppose You have three gambles that are acceptable to You:



Question: What is the smallest, most conservative, coherent model that corresponds to Your dispositions?

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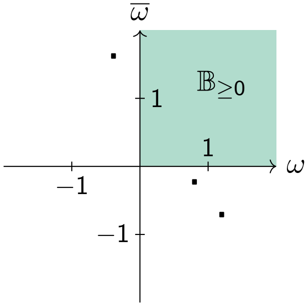

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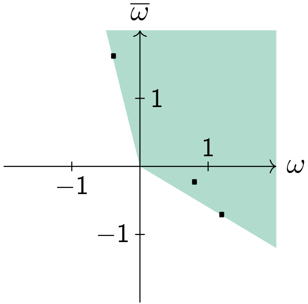
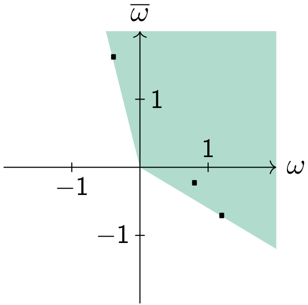


Question: What is the smallest, most conservative, coherent model that corresponds to Your dispositions?

Inference through natural extension

The natural extension $\mathcal{E}_A$ of Your elicited set of accepted gambles A wrt $\mathcal{X}$ is

$$\mathcal{E}_A := \left\{ \sum_{k=1}^{n} \lambda_k f_k \;:\; f_k \in A \cup \mathbb{B}_{\geq 0},\ \lambda_k \geq 0,\ n > 0 \right\}$$



Conditional Lower Previsions

Let's introduce **conditional lower previsions** for unbounded random variables defined over

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Let $\wp^+(\mathcal{X})$ be the **powerset** of $\mathcal{X}$ excluding the empty set.

Now consider a set of **acceptable unbounded gambles** $D \subseteq \mathbb{G}$.

Conditional Lower Previsions

Define for $\mathcal{D} \subseteq \mathbb{G}$ two functionals on $\mathbb{G} \times \wp^+$:

The **conditional lower prevision** $\mathrm{lpr}(\mathcal{D})(| \cdot) : \mathbb{G} \times \wp^+ \to \mathbb{R}^*$:

$$\mathrm{lpr}(\mathcal{D})(f \mid E) \; := \; \{\alpha \in \mathbb{R}^* : (f - \alpha)I_E \in \mathcal{D}\}$$

The **conditional upper prevision** $\mathrm{upr}(\mathcal{D})(\cdot \mid \cdot) : \mathbb{G} \times \wp^+ \to \mathbb{R}^*$:

$$\mathrm{upr}(\mathcal{D})(f \mid E) \; := \; \{\alpha \in \mathbb{R}^* : (\alpha - f)I_E \in \mathcal{D}\}$$

for any gamble f on $\mathcal{X}$ and any event $E \in \wp^+$.

Properties of conditional lower previsions on gambles

Theorem (Williams 1975, Troffaes and De Cooman 2014)

Let $\mathrm{lpr}(\mathcal{D})(\cdot \mid \cdot)$ be the conditional lower prevision on $\mathcal{D}$. Then, for all gambles f, g, all non-negative real numbers λ, all real numbers α, and all non-empty events A and B such that $A \subseteq B$, we have

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$\mathrm{lpr}(\mathcal{D})(f + g \mid A) \geq \mathrm{lpr}(\mathcal{D})(f \mid A) + \mathrm{lpr}(\mathcal{D})(g \mid A)$
whenever the right-hand side is well defined

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$$\mathrm{lpr}(\mathcal{D})(\lambda f \mid A) = \lambda \mathrm{lpr}(\mathcal{D})(f \mid A).$$

clp.4 **Bayes' rule**:
$$\mathrm{lpr}(\mathcal{D})(f - \alpha)I_A \mid B) \begin{cases} \geq 0 & \text{if } \alpha < \mathrm{lpr}(\mathcal{D})(f \mid A) \\ \leq 0 & \text{if } \alpha > \mathrm{lpr}(\mathcal{D})(f \mid A). \end{cases}$$

Representing conditional lower previsions

Theorem (Walley 1991, Troffaes and De Cooman 2014)

A $\mathbb{R}^*$-valued functional $\underline{P}(\cdot \mid \cdot)$ on $\mathbb{G} \times \wp^+$ is a conditional lower prevision iff there is some set of acceptable gambles $\mathcal{D}$ such that

- $\underline{P}(\cdot \mid \cdot) = \mathrm{lpr}(\mathcal{D})(\cdot \mid \cdot)$, and
- $\underline{P}(\cdot \mid \cdot)$ satisfies clp1 – clp4.

Example

$\underline{P}(f \mid A) = -\infty$ says that, given A, You are not willing to buy f at any price.

Coherence

Coherence for conditional lower previsions can be purchased in each of the following (equivalent) ways:

1. There is some coherent set of acceptable gambles $\mathcal{D}$ such that $\mathrm{lpr}(\mathcal{D})(\cdot \mid \cdot)$ is an extension of $\underline{P}(\cdot \mid \cdot)$

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2. $\underline{P}(\cdot \mid \cdot)$ avoids sure loss and is a restriction of the **natural extension** of $\underline{P}(\cdot \mid \cdot)$.

Two ways of defining lower previsions w/out credal sets

Directly

- Define $\underline{P}(f)$ directly as Your supremum acceptable buying price for f
- $\underline{P}(f)$ is the highest price s s.t. for any $t < s$, You accept to pay t before observing X on the promise that You will receive $f(x)$ upon observing the event $X = x$.
- *Thus*, You are only required to consider whether you accept bounded gambles of type $f - \alpha$.
- *Conditioning is complicated*

Via Gambles

- Announce Your acceptable bounded gambles
- Enforce coherence conditions
- *Then* infer Your lower $\underline{P}(f)$ and upper $\overline{P}(f)$ previsions for any bounded gamble f.
- *Conditioning is clean*

Three goals

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Summary

- IP via sets of acceptable gambles is ready for general use
- It meets the minimal acceptable benchmarks
  - Operationalizable
  - Inference: closure, conditionalization, marginalization
  - Unifies different accounts: propositional logic, Bayesian probabilities, linear previsions, and lower previsions are all special cases.
- Next: Objections to IP.

Alexander von Humboldt
Stiftung / Foundation

Rationality Criteria for Sets of Acceptable Gambles

For all bounded-gambles f and g on $\mathcal{X}$, and all non-negative real numbers λ,

A1. **Avoid Partial Loss** - If $f < 0$, then $f \notin \mathcal{D}$.

You should not accept any gamble you cannot win.

A2. **Accept Partial Gain** - If $f \geq 0$, then $f \in \mathcal{D}$.

You should accept any gamble you cannot lose.

A3. **Scale Invariance** - If $f \in \mathcal{D}$, then $\lambda f \in \mathcal{D}$.

If you accept gamble f, you should accept λf.

A4. **Combination** - If $f \in \mathcal{D}$ and $g \in \mathcal{D}$, then $f + g \in \mathcal{D}$.

If you accept f and accept g, you should accept $f + g$.

A5. **Monotonicity** - If $f \in \mathcal{D}$ and $g \geq f$, then $g \in \mathcal{D}$.

Any gamble dominating an acceptable gamble is acceptable.

NB: A5 is a theorem. It follows from A2 and A4.

Information ordering

Suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ are the sets of acceptable gambles for subject$_1$ and subject$_2$, respectively.

Information ordering

Suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ are the sets of acceptable gambles for subject$_1$ and subject$_2$, respectively.

If $\mathcal{D}_1 \subseteq \mathcal{D}_2$, then subject$_2$ will accept (at least) all the bounded gambles that subject$_1$ does.

We may interpret '$\subseteq$' between sets of acceptable bounded gambles as:

- '*is at most as informative as*'
- '*is at most as committal as*'
- '*is at least as conservative as*'

Extensions

Suppose $\mathcal{A}$ is Your set of acceptable bounded gambles. Your **assessment** $\mathcal{A}$ is (likely) to be finite. Suppose we **extend** $\mathcal{A}$ by the following set of bounded gambles:

$$\mathcal{G}_{\mathcal{A}} := \left\{ g + \sum_{k=1}^{n} \lambda_k f_k \mid g \geq 0, n \in \mathbb{N}, f_k \in \mathbb{R}_{\geq 0}, k = 1, \ldots, n \right\}$$

where $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers.

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where $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers.

$\mathcal{G}_{\mathcal{A}}$ is the smallest set of bounded gambles that includes $\mathcal{A}$ and satisfies A2 - A4.

Consistency

A set $\mathcal{A}$ of acceptable bounded gambles is **consistent** or **avoids partial loss** if one (or both) of the equivalent conditions is satisfied:

A. $\mathcal{A}$ is included in some coherent set of acceptable bounded gambles:

$$\mathcal{D} \in \mathbb{D}_b \mid \mathcal{A} \subseteq \mathcal{D}\} \neq \emptyset.$$

B. For all $n \in \mathbb{N}$, non-negative $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, and bounded gambles $f_1, \ldots, f_n \in \mathcal{A}$:

$$\sum_{k=1}^{n} \lambda_k f_k \not< 0.$$

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$$\sum_{k=1}^{n} \lambda_k f_k \nless 0.$$

NB: Condition B explain why consistency is also called **avoiding partial loss**: no non-negative linear combination of bounded gambles in $\mathcal{A}$ should produce a partial loss.

Closure

Proposition: Given a non-empty class $\mathcal{D}_i, i \in I$ of sets of acceptable bounded gambles, if all $\mathcal{D}_i$ are coherent, then so is their intersection $\bigcap_{i \in I} \mathcal{D}_i$.

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For any assessment $\mathcal{A}$, consider the set $\{\mathcal{D} \in \mathbb{D}_b \mid \mathcal{A} \subseteq \mathcal{D}\}$ of all coherent sets of acceptable bounded gambles including $\mathcal{A}$. Define the intersection of this set to be the **closure** $\mathrm{Cl}_{\mathbb{D}_b}(\mathcal{A})$ of $\mathcal{A}$:

$$\mathrm{Cl}_{\mathbb{D}_b}(\mathcal{A}) := \bigcap \{\mathcal{D} \in \mathbb{D}_b : \mathcal{A} \subseteq \mathcal{D}\}$$

NB: the intersection of the empty set is the set of all bounded gambles: $\bigcap \emptyset := \mathbb{B}$.

Properties of Closure

Let $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$ be sets of acceptable bounded gambles. Then the following hold:

1. $\mathcal{A} \subseteq \mathsf{Cl}_{\mathbb{D}_b}(\mathcal{A})$

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4. If $\mathcal{A} \subseteq \mathbb{B}_{\geq 0}$ then $\mathsf{Cl}_{\mathbb{D}_b}(\mathcal{A}) = \mathbb{B}_{\geq 0}$.

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4. If $\mathcal{A} \subseteq \mathbb{B}_{\geq 0}$ then $\mathsf{Cl}_{\mathbb{D}_b}(\mathcal{A}) = \mathbb{B}_{\geq 0}$.
5. $\mathcal{A}$ is consistent iff $\mathsf{Cl}_{\mathbb{D}_b}(\mathcal{A}) \neq \mathbb{B}$.

Properties of Closure

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4. If $\mathcal{A} \subseteq \mathbb{B}_{\geq 0}$ then $\mathrm{Cl}_{\mathbb{D}_b}(\mathcal{A}) = \mathbb{B}_{\geq 0}$.
5. $\mathcal{A}$ is consistent iff $\mathrm{Cl}_{\mathbb{D}_b}(\mathcal{A}) \neq \mathbb{B}$.
6. $\mathcal{A}$ is a coherent set of acceptable bounded gambles iff it is consistent and $\mathcal{A} = \mathrm{Cl}_{\mathbb{D}_b}(\mathcal{A})$.

NB: Any $\mathcal{P}(\mathbb{B}) - \mathcal{P}(\mathbb{B})$-map that satisfies (1) - (3) is a **closure operator**; (4) says that the corresponding **closed subsets** of $\mathcal{P}(\mathbb{B})$ different from $\mathbb{B}$ are exactly the coherent sets of acceptable bounded gambles.

Natural Extension

Theorem (Natural Extension) If the set $\mathcal{A}$ of acceptable bounded gambles is consistent, then there is a smallest coherent set of acceptable bounded gambles that includes $\mathcal{A}$. It is given by

$$\mathrm{Cl}_{\mathbb{D}_b}(\mathcal{A}) = \mathcal{G}_{\mathcal{A}} = \left\{ g + \sum_{k=1}^{n} \lambda_k f_k \mid g \geq 0, n \in \mathbb{N}, f_k \in \mathbb{R}_{\geq 0}, k = 1, \ldots, n \right\}$$

$$= \left\{ h \in \mathbb{B} \mid h \geq \sum_{k=1}^{n} \lambda_k f_k \text{for some } n \in \mathbb{N}, f_k \in \mathcal{A}, \lambda_k \in \mathbb{R}_{\geq 0} \right\}$$

and it is called the **natural extension** of $\mathcal{A}$.

Inference

Applying the **closure operator**, $\mathrm{Cl}_{\mathbb{D}_b}$, or (equivalently) **natural extension**, to a consistent assessment $\mathcal{A}$ adds those bounded gambles to $\mathcal{A}$ that can be obtained from bounded gambles in $\mathcal{A}$ using the **production rules** A2 - A4 and no other gambles.

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Thus, **natural extension** is a conservative inference mechanism: it picks the **smallest** coherent set of bounded gambles (with respect to some $\mathcal{A}$ that satisfy A2 - A4.

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Thus, **natural extension** is a conservative inference mechanism: it picks the **smallest** coherent set of bounded gambles (with respect to some $\mathcal{A}$ that satisfy A2 - A4.

Consistency then ensures A1, namely that the production rules will not lead to negative bounded gambles and therefore will not lead to partial loss.

Propositional Logic as a Special Case of Natural Extensions

The inference mechanism of natural extension subsumes that of classical propositional logic.

Propositional Logic as a Special Case of Natural Extensions

Setup:

- An **event** is a subset of the possible values $\mathcal{X}$ of X.
- The **indicator** I_E of an event E is the bounded gamble that gives one if the actual value x of X belongs to E and zero otherwise.

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- If You **accept a proposition** p, then You accept the bounded gambles $I_{E_p} - 1 + \epsilon$ for all $\epsilon > 0$.

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- If You **accept a proposition** p, then You accept the bounded gambles $I_{E_p} - 1 + \epsilon$ for all $\epsilon > 0$.
- Since,

$$I_{E_p}(x) - 1 + \epsilon = \begin{cases} \epsilon & \text{if } x \in E_p \\ \epsilon - 1 & \text{if } x \notin E_p, \end{cases}$$

  You are willing to bet on the occurrence of E_p at **any** odds $\frac{1-\epsilon}{\epsilon}$, i.e., at any rate strictly less than 1.
- For any rate $(1-\epsilon)/\epsilon < 1$, You are practically certain that E_p occurs.

Propositional Logic as a Special Case of Natural Extensions

Logical Production Rules

- **Modus Ponens:** Suppose You accept both p and $p \Rightarrow q$. Then, $E_p \subseteq E_q$, so,

$$I_{E_p}(x) - 1 + \epsilon \;\leq\; I_{E_q}(x) - 1 + \epsilon.$$

  From Monotonicity (A5), it follows that You should accept E_q.
- Therefore, the logical inference rule modus ponens is included in the inference mechanism for coherent sets of acceptable gambles.

Propositional Logic as a Special Case of Natural Extensions

Logical Production Rules

- **Conjunction:** Suppose You accept both p and q.

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Logical Production Rules

- **Conjunction:** Suppose You accept both p and q.

  Then, You accept the bounded gambles $I_{E_p}(x) - 1 + \epsilon$ and $I_{E_q}(x) - 1 + \epsilon$, and by A4 You also accept the bounded gambles $I_{E_p}(x) + I_{E_q}(x) - 2 + 2\epsilon$, for all $\epsilon > 0$.

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  Then, You also accept the conjunction $p \wedge q$ by A5, since $E_{p \wedge q} = E_p \cap E_q$ and

  $$I_{E_p}(x) + I_{E_q}(x) - 1 \ \leq \ I_{E_p \cap E_q}(x) \ = \ \min\{I_{E_p}, I_{E_q}\}.$$

- Therefore, the logical rule of conjunction is included in the inference mechanism for coherent sets of acceptable bounded gambles.

Propositional Logic as a Special Case of Natural Extensions

Logical Production Rules

- **Logical Non-Contradiction:** Suppose You accept both p and $\neg p$.

Propositional Logic as a Special Case of Natural Extensions

Logical Production Rules

- **Logical Non-Contradiction:** Suppose You accept both p and $\neg p$.

  The, You would accept both $I_{E_p}(x) - 1 + \epsilon$ and $I_{E_{\neg p}}(x) - 1 + \epsilon$, and by A4, You would also accept the bounded gambles

$$I_{E_p}(x) - 1 + \epsilon + I_{E_{\neg p}}(x) - 1 + \epsilon \ = \ -1 + 2\epsilon$$

  for all $\epsilon > 0$, since $I_{E_p}(x) + I_{E_{\neg p}}(x) = 1$. But this contradicts A1, avoiding partial loss.

- Therefore, logical contradiction is prohibited by the inference mechanism for coherent sets of acceptable bounded gambles.

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