Open problems and recent results on causal completeness of probabilistic theories

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Structure

- Informal motivation of the problem of causal closedness:
  Reichenbach's Common Cause Principle
- Causal closedness of classical probability spaces (notion + propositions)
- Causal closedness – quantum probability spaces (notion + proposition)
- Spacelike correlations predicted by quantum field theory
- Local common causes in quantum field theory (notion of causal closedness + unsatisfactory proposition)

Motivation

Reichenbach's Common Cause Principle

compact formulation:
No correlation without causation

Explicitly:

If two events A, B are probabilistically correlated, then either there is a causal connection between A and B that is responsible for the correlation or, if A and B are causally independent, $R_{ind}(A, B)$, then there is a third event C, a (Reichenbachian) common cause which brings about the correlation

The problem

If

Reichenbach's Common Cause Principle
is assumed to be true/valid

$\Downarrow$

Can our probabilistic theories be
causally closed

$\Updownarrow$

Providing causal explanation of the correlations they predict
in terms of
either
describing the causal link between the correlated entities
or
by providing common causes of the correlations?

Main message

- Causal closedness of classical probability theories is
  - non-trivial
  - not impossible
  - possibly not typical
- Causal closedness of non-classical (quantum) probability theories
  - seems even more non-trivial
  - not investigated extensively
  - many open problems
- Causal closedness of relativistic quantum field theory
  - can be defined precisely in different ways
  - is an open problem
  - only unsatisfactory result is known

Reichenbach's notion of common cause

Definition:

$(\mathcal{S}, p)$ classical probability space

$C \in \mathcal{S}$ is a common cause of the correlation

$$p(A \cap B) > p(A)p(B)$$

if

$$\begin{aligned} p(A \cap B|C) &= p(A|C)p(B|C) \\ p(A \cap B|C^{\perp}) &= p(A|C^{\perp})p(B|C^{\perp}) \\ p(A|C) &> p(A|C^{\perp}) \\ p(B|C) &> p(B|C^{\perp}) \end{aligned}$$

Causal completeness

In harmony with Reichenbach's Common Cause Principle:

Definition:

$(\mathcal{S}, p)$ is causally closed (complete)
with respect to a causal independence relation R_{ind} on $\mathcal{S}$
if $\mathcal{S}$ contains a common cause of every correlation between elements A, B such that $R_{ind}(A, B)$ holds

Danger of trivialization

One can make every probability space $(\mathcal{S}, p)$ causally complete

by defining:
$R_{ind}(A, B)$ holds
whenever A and B are correlated
but there is no common cause of this correlation in $\mathcal{S}$

Trivial end of story of causal completeness?

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Trivial end of story of causal completeness?

Too cheap!
We need a disciplined definition of causal independence!

Condition on R_{ind}

Intuition: causal independence of A and B should imply that from the presence or absence of A one should not be able to infer either the occurrence or non-occurrence of B, and conversely: presence or absence of B should not entail occurrence or non-occurrence of A.

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Definition:
$A, B \in \mathcal{S}$ are called logically independent if

$$A \not\subseteq B, \quad A^{\perp} \not\subseteq B \quad , \quad A \not\subseteq B^{\perp}, \quad A^{\perp} \not\subseteq B^{\perp}$$

$$B \not\subseteq A, \quad B^{\perp} \not\subseteq A \quad , \quad B \not\subseteq A^{\perp}, \quad B^{\perp} \not\subseteq A^{\perp}$$

Logical independence

Definition:

Two Boolean subalgebras $\mathcal{L}_1, \mathcal{L}_2$ of $\mathcal{S}$ are called logically independent if any $0 \neq A \in \mathcal{L}_1$ and $0 \neq B \in \mathcal{L}_2$ are logically independent

equivalently:

if

$$A \cap B \neq 0$$

for $0 \neq A \in \mathcal{L}_1 \qquad 0 \neq B \in \mathcal{L}_2$

Causal closedness & logical independence

Definition:

$(\mathcal{S}, p)$ is causally closed with respect to logically independent Boolean sublattices $\mathcal{L}_1, \mathcal{L}_2$ if $\mathcal{S}$ contains a common cause of every correlation between $A \in \mathcal{L}_1$ and $B \in \mathcal{L}_2$

Causal closedness

Proposition:

$(\mathcal{S}_5, p_u)$ is non-trivially causally closed with respect to every pair of logically independent Boolean subalgebras (p_u = uniform probability on atoms of $\mathcal{S}_5$)

$\uparrow$

Surprising!

Very strong causal completeness !!

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Very strong causal completeness !!

The very strong causal completeness of $(\mathcal{S}_5, p_u)$ is truly exceptional:

Proposition:

If $(\mathcal{S}_n, p)$ is not $(\mathcal{S}_5, p_u)$ then $(\mathcal{S}_n, p)$ is not non-trivially causally closed with respect to every pair of logically independent Boolean subalgebras

Causal closedness

Proposition:

For any $n \geq 5$, if $\mathcal{S}_n$ is a finite Boolean algebra generated by n atoms, then there exists a probability measure p on $\mathcal{S}_n$ and there exist two logically independent Boolean subalgebras $\mathcal{L}_1, \mathcal{L}_2$ of $\mathcal{S}_n$ such that $(\mathcal{S}_n, p)$ is causally closed with respect to $(\mathcal{L}_1, \mathcal{L}_2)$.

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- It is not known how typical or untypical common cause completeness is (with respect to an R_{ind} stronger than logical independence) in finite probability spaces
- There is no straightforward test to tell if a probability space is causally complete

Comm. cause cl.ness in ∞ prob. sp.

Proposition:

Nonatomic probability spaces are common cause closed: containing a common cause of every correlation they predict

Comm. cause cl.ness in ∞ prob. sp.

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Nonatomic probability spaces are common cause closed: containing a common cause of every correlation they predict

Definition:

$(\mathcal{S}, p)$ is a nonatomic probability space if for A with $p(A) > 0$ there exists $0 \neq B \subset A$ with $0 \neq p(B) < p(A)$

Example: $([0,1], \mathcal{B}([0,1]), p)$

p = Lebesgue measure

$\mathcal{B}([0,1])$ = Lebesgue measurable sets

Non-commutative probability space

Replace $(\mathcal{S}, p)$ by $(\mathcal{L}, \phi)$

$(\mathcal{L}, \wedge, \vee, \perp)$ = non-distributive orthocomplemented (σ-) lattice
ϕ = (σ-) additive bounded measure on $\mathcal{L}$

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Special cases:

- $\mathcal{L} = \mathcal{P}(\mathcal{H})$ lattice of projections on a Hilbert space
  ϕ = a (quantum) state
- $\mathcal{L} = \mathcal{P}(\mathcal{N})$ lattice of projections of a von Neumann algebra
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$A, B \in \mathcal{L}$ compatible elements are correlated in ϕ if

$$\phi(A \wedge B) > \phi(A)\phi(B)$$

Comm. cause in non-comm. prob. space

Definition:

$C \in \mathcal{L}$ is a common cause of the correlation if
C is compatible with both A and B and

$$\frac{\phi(A \wedge B \wedge C)}{\phi(C)} = \frac{\phi(A \wedge C)}{\phi(C)} \frac{\phi(B \wedge C)}{\phi(C)}$$

$$\frac{\phi(A \wedge B \wedge C^{\perp})}{\phi(C^{\perp})} = \frac{\phi(A \wedge C^{\perp})}{\phi(C^{\perp})} \frac{\phi(B \wedge C^{\perp})}{\phi(C^{\perp})}$$

$$\frac{\phi(A \wedge C)}{\phi(C)} > \frac{\phi(A \wedge C^{\perp})}{\phi(C^{\perp})}$$

$$\frac{\phi(B \wedge C)}{\phi(C)} > \frac{\phi(B \wedge C^{\perp})}{\phi(C^{\perp})}$$

Non-comm. comm. cause completeness

Definition of

- common cause (in)completeness of $(\mathcal{L}, \phi)$
- causal independence relation R_{ind} on $\mathcal{L}$
- logical independence
  and
  logically independent sublattices of $\mathcal{L}$
- causal closedness of $(\mathcal{L}, \phi)$ with respect to R_{ind}

in complete analogy with the Boolean case

Problems

Problem 1 : Under what conditions on $(\mathcal{L}, \phi)$ and R_{ind} is the non-classical probability space $(\mathcal{L}, \phi)$ causally closed with respect to R_{ind}?

Problem 1.1 : How about causal completeness of the specific quantum probability space $(\mathcal{H}, \mathcal{P}(\mathcal{H}), \phi)$ (with respect to some R_{ind})?

Problem 1.2 : How about causal closedness of non-commutative probability spaces $(\mathcal{P}(\mathcal{N}), \phi)$ determined by a non-commutative von Neumann algebra $\mathcal{N}$ and a normal state ϕ on $\mathcal{N}$? What role does the (Murray-von Neumann) type of $\mathcal{N}$ play in causal closedness?

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Mainly open problems, only result known:

Proposition (Kitajima, 2007)

If $\mathcal{L}$ is an atomless, complete, orthomodular lattice then it is causally closed with respect to every pair of logically independent sublattices

Local common causes in AQFT

(Algebraic) Relativistic Quantum Field Theory (AQFT)

$\|$

$\{\mathcal{A}, \mathcal{A}(V), V \subset M\}$ local net of observables

predicts correlations

between spacelike separated observables

spacelike related entities cannot stand in causal relation

$\Downarrow$

causal completeness of AQFT requires

existence in AQFT of common causes

of spacelike correlations

but

all observables (projections) in AQFT must be local

$\Downarrow$

common causes must be local

$\Downarrow$

where should they be localized?

Possible regions of locality

$BLC(V)$: union of the backward light cones of every point in V
For spacelike V_1, V_2 three possible causally non-independent regions:

$$\begin{aligned} wpast(V_1, V_2) &\equiv (BLC(V_1) \setminus V_1) \cup (BLC(V_2) \setminus V_2) \\ cpast(V_1, V_2) &\equiv (BLC(V_1) \setminus V_1) \cap (BLC(V_2) \setminus V_2) \\ spast(V_1, V_2) &\equiv \cap_{x \in V_1 \cup V_2} BLC(x) \end{aligned}$$

- $spast(V_1, V_2)$ consists of spacetime points each of which can causally influence every point in both V_1 and V_2
- $cpast(V_1, V_2)$ consists of spacetime points each of which can causally influence at least some point in both V_1 and V_2
- $wpast(V_1, V_2)$ consists of spacetime points each of which can causally influence at least some point in either V_1 or V_2

Local common cause in AQFT

Definition If $(\mathcal{A}(V_1), \mathcal{A}(V_2), \phi)$ is a local system in AQFT such that

$$\phi(A \wedge B) > \phi(A)\phi(B) \tag{1}$$

for some $A \in \mathcal{A}(V_1), B \in \mathcal{A}(V_2)$ and there exists a projection C in a von Neumann algebra $\mathcal{A}(V)$ which is a common cause of the correlation (1), then the local system $(\mathcal{A}(V_1), \mathcal{A}(V_2), \phi)$ is said to satisfy the

Weak Common Cause Principle	if	V	$\subseteq$	$wpast(V_1, V_2)$
Common Cause Principle	if	V	$\subseteq$	$cpast(V_1, V_2)$
Strong Common Cause Principle	if	V	$\subseteq$	$spast(V_1, V_2)$

Def. of causal completeness of AQFT

Definition AQFT is causally complete (Reichenbach's Common Cause Principle holds in AQFT) (respectively in the weak or strong sense) iff for every pair of spacelike separated convex spacetime regions V_1, V_2 and every normal state ϕ, the local system $(\mathcal{A}(V_1), \mathcal{A}(V_2), \phi)$ satisfies the Common Cause Principle (respectively in the weak or strong sense)

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Problem

Is AQFT causally complete?

Prop. on causal closedness of AQFT

Proposition 1 If the net $\{\mathcal{A}(V)\}$ satisfies the standard conditions including local primitive causality, then every local system $(\mathcal{A}(V_1), \mathcal{A}(V_2), \phi)$ with V_1, V_2 spacelike separated double cones and with a locally normal and locally faithful state ϕ satisfies the Weak Common Cause Principle

"AQFT is weakly causally complete"

Prop. on causal closedness of AQFT

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Proposition 2 AQFT is (trivially) not strongly causally complete

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"AQFT is weakly causally complete"

Proposition 2 AQFT is (trivially) not strongly causally complete

It is not known if AQFT is casually complete

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stronger than weak, weaker than strong

Summary I

- Causal completeness is a natural feature to ask of probabilistic theories if one assumes Reichenbach's Common Cause Principle to be valid
- Nonatomic classical probability spaces are common cause closed
- Finite probability theories may or may not be causally closed with respect to a causal independence relation stronger than logical independence

Summary II

- Common cause completeness of non-commutative probability spaces with respect to R_{ind} can be defined, little is known about causal closedness of non-classical probability spaces
- AQFT predicts correlations between observables belonging to algebras associated with spacelike separated (hence causally independent) spacetime regions
- Locality of a common cause of spacelike correlations in AQFT can be specified in different ways
- It is an open problem whether AQFT contains suitable localized common causes of the spacelike correlations it predicts

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