The method of arbitrary functions (*MAF*):

Causality and probability

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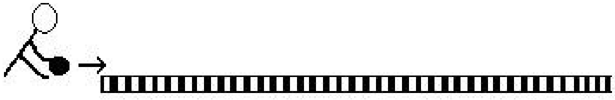
Idea of (MAF)

$$\underbrace{\begin{array}{ll} (1) & \text{mechanistical (or causal) features} \\ (2) & \text{symmetries} \\ (3) & \text{a general probability assumption} \end{array}}$$

$$\Downarrow \begin{array}{l}\text{Explain/} \\ \text{Infer}\end{array}$$

Unique initial probability attributions

Stoss-Spiel (Kries)



W : the event of **landing on a white segment**

B : the event of **landing on a black segment**

Claim : $P(W) = P(B) = 0.5$

1. All segments are of same size
2. Colours of segments alternate (rapidely and in greate number)
3. There is a continous probability distribution ϕ on points of the Strip.

$$B = \bigcup_{i \in I} b_i \quad , \quad W = \bigcup_{j \in J} w_j$$

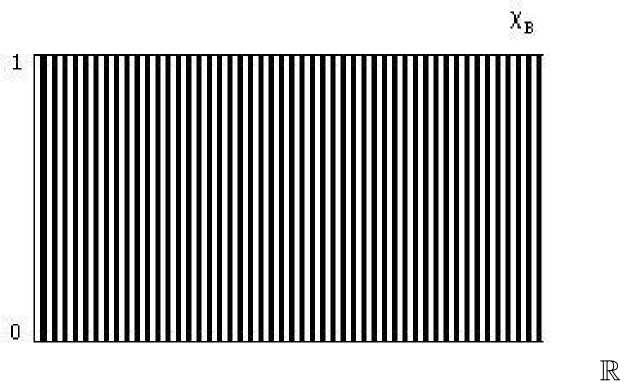
$$p(b_i) = \int_{b_i} \phi(x)\ dx \quad , p(w_j) = \int_{w_j} \phi(x)\ dx$$

$$P(B) = \sum_{i \in I} \int_{b_i} \phi(x)\ dx \quad , \quad P(W) = \sum_{j \in J} \int_{w_j} \phi(x)\ dx$$

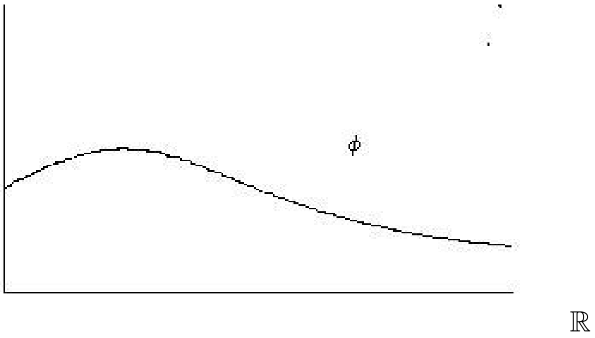
$\mathbb{R}$

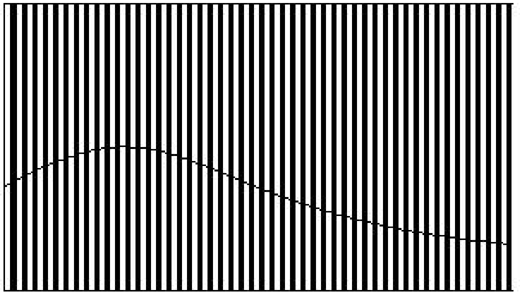
$$\chi_B : \mathbb{R} \longrightarrow \{0,1\}$$

$$\chi_B(\omega) = \begin{cases} 1 & \text{if} \quad \omega \text{ is black} \\ 0 & \text{else.} \end{cases}$$

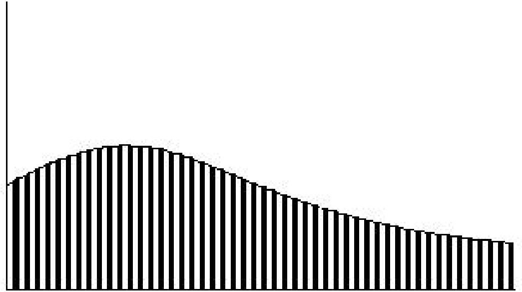


$$\phi : \mathbb{R} \rightarrow [0,1]$$





$\mathbb{R}$



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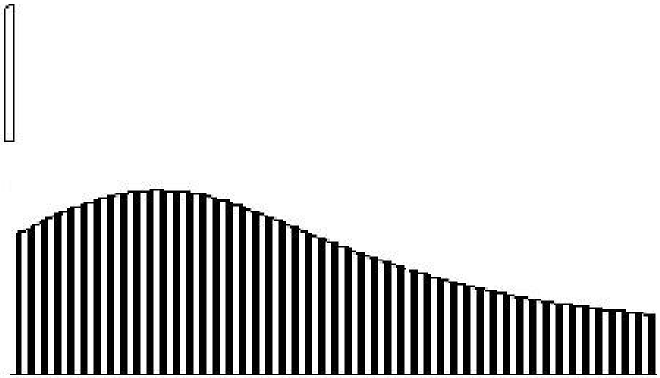
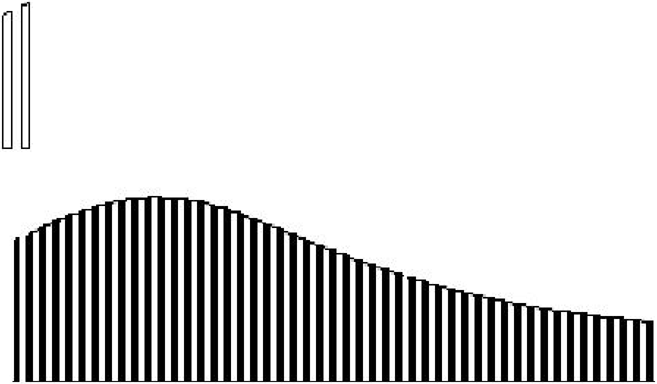
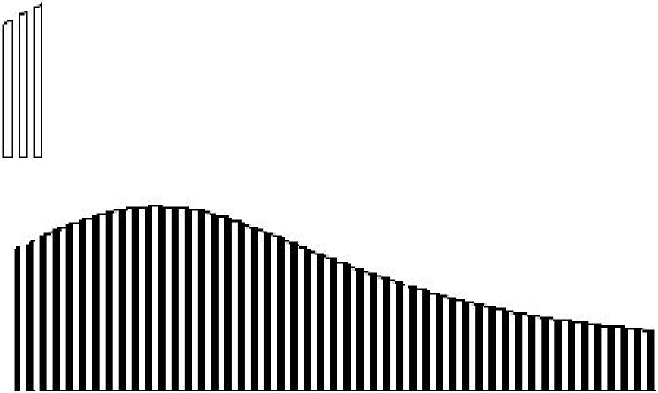
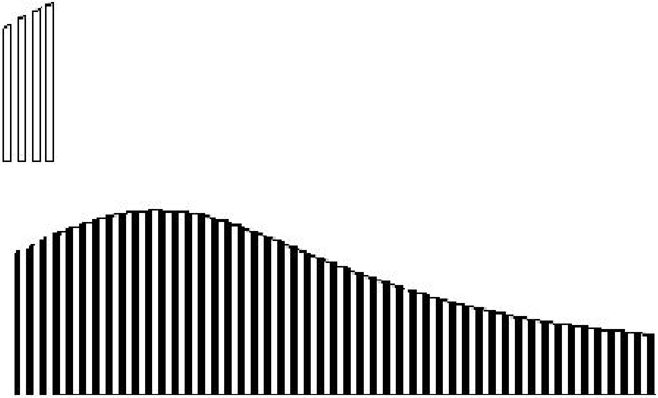
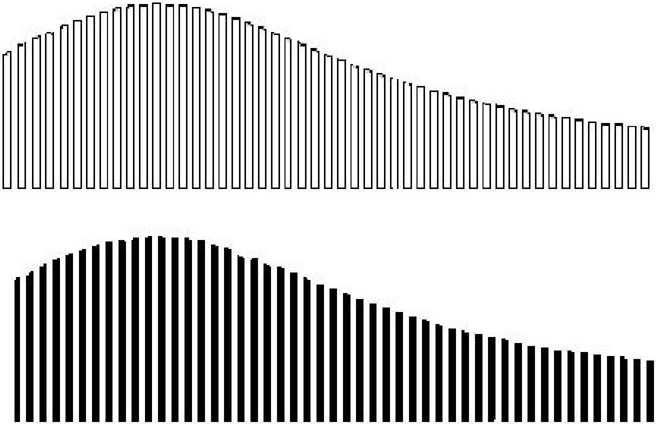
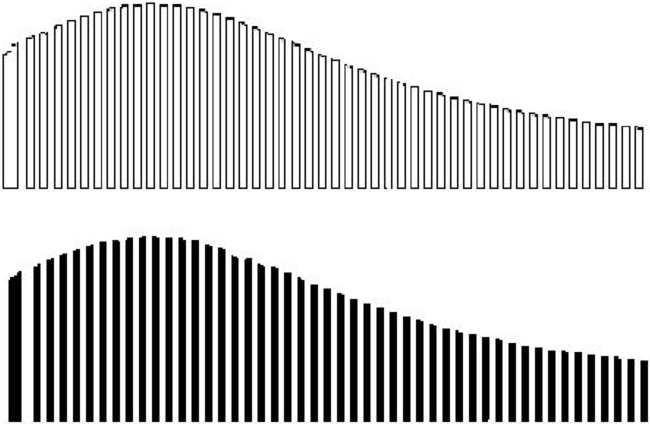


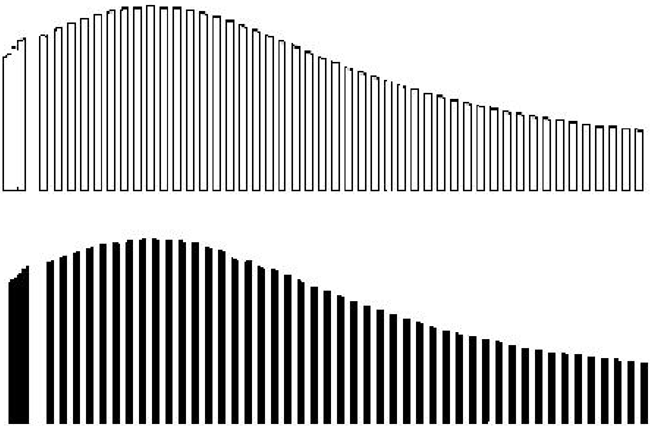


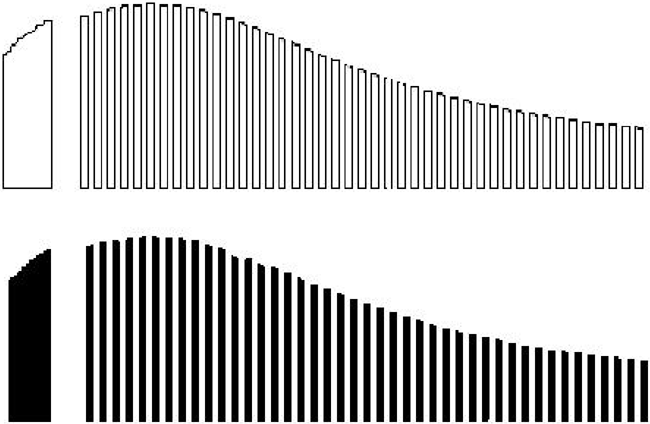


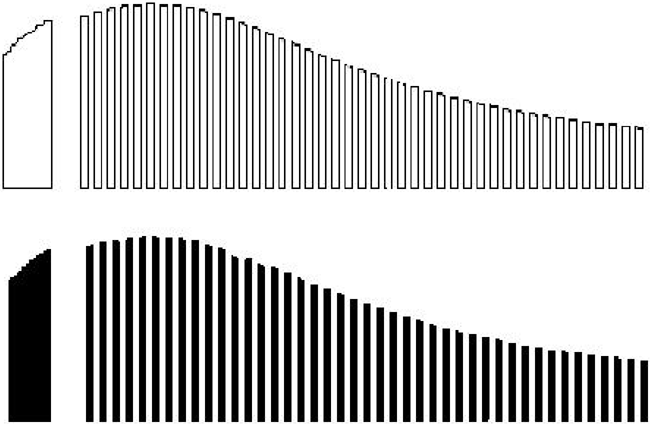


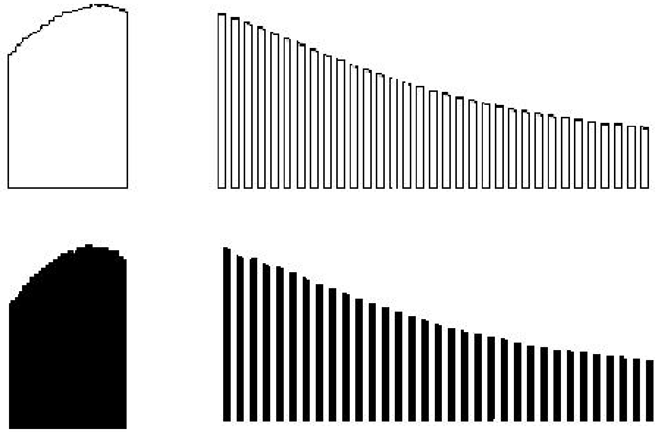


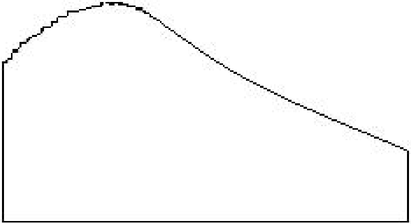




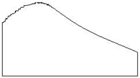








$\approx$



$$P(W) = \int_{\bigcup W} \phi(\omega)\ d\omega$$

$$\approx$$

$$P(B) = \int_{\bigcup B} \phi(\omega)\ d\omega$$

Theorem (Poincare-Reichenbach)

If

1. $\mathbb{R}$ *is* **adequately partitionned**
   - *into intervals*
   - *each interval (every point of each interval) is of one of two colours (say red and black)*
   - *all intervals are of equal size*
   - *colours of intervals alternate*

   and

2. *there exists* $\phi : \mathbb{R} \rightarrow [0, 1]$ *a* **continuous probability function** *(with finitely normed integral*[a]*),*

then $p(R) \approx p(B)$*, and* $p(R) = p(B)$ *if interval size tends to* 0*.*

[a] $\int_{\mathbb{R}} \phi(\omega)\ d\omega = 1$.

where $p(R) = \int \chi_R(x)\phi(x)\ dx$

Adequately partitionned

Let S be a space, $\chi : S \to \{0,1\}$ a two colouring on S.
N_S is an **adequate partition** of S, if

1. it is a partition of S (i.e. $\bigcup N_S = S$ and $\forall I, J \in N_S, I \cap J = \emptyset$)
2. All $I \in N_S$ are intervals (cells) of S (of same type)
3. For each $I \in N_S$ there exists $c \in \{0,1\}$, $\chi(I) = c$
4. if I, J are close then $\chi(I) \neq \chi(J)$
5. Forall $I, J \in N_S$, $m(I) = m(J)$

N_S is of the form $S_0/\sim_0 \bigcup S_1/\sim_1$, where S_c are classes of points of same colour and the classes of $S_c/\sim c$ are the biggest intervals (or cells) where no colour change occures.

Results for initial distributions

Let $N = N_S$ be a net on a space S (for resulttype R).

- (MAF) **defines a unique initial probability assignment** P_N for R. If $\phi, \phi' : S \to [0,1]$ are continuous, then

$$P_N^{\phi}(R) = P_N^{\phi'}(R) =: P_N$$

Let $\psi : S \to S'$ be a bijection

- If N, N' are $\psi-$isomorphic nets (same comparison ratios, etc.) on S, resp. S' for resulttype R respectively $\psi[R]$, then

$$P_N(R) = P_{N'}(\psi[R])$$

What justifies (empirically)...

1. colour alternation ?
2. local comparability ?
3. continuity of distribution ?

Representation of the mechanisme

- The **mechanisme** of a chance-set up is characterised by a **deterministic** (one-to-one) **function**

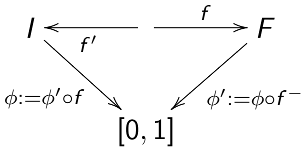
$$f : I \rightarrow F$$

$$\overrightarrow{x} \mapsto \overrightarrow{y}$$

  where I are the **initial conditions** and F the **final conditions**. ($I \subseteq \mathbb{R}^n, F \subseteq \mathbb{R}^m$ for $n, m \in \mathbb{N}$. Here $n, m = 1$). Resulttypes (colours) are subsetets (union of disjoint intervals) of F.

- **Additionnally Causal requierement :**
  if $n > 1$, then variables x_i x_j $i, j \in n$ need to be **mutually independent**. (i.e. big changes in x_i do not essentially change the variation of x_j (and vice versa).)

Some results



Let f be one-to-one. If f and ϕ are continuous (resp. ϕ') then $\phi' = \phi \circ f^{-}$ (resp. $\phi = \phi' \circ f$) is continuous.

Consequence : It comes down to the same to suppose a continuous distribution on initial or final conditions (as long as f is one to one and continuous).

Some results

Let $f : S \rightarrow S'$ be a (strictly increasing or decreasing) differentiable function

- If N is a net on S then $N' = f[N] = \{f[I] : I \in N\}$ is a net on S' and isomorphic to N (and vice versa)
- (MAF) is **invariant for differentiable functions**. I.e. if f is differentiable, N a net on S then

$$P_N(R) = P_{f[N]}(f[R])$$

Consequence : (MA) is *invariant for equivalent physical problems*. It is irrelevant whether (MA) is applied to initial or final conditions (or any other parameter which depends on the others by a differentiable transformation).

Colour alternation : small variations in causes - big variations in effects

f is such that for a little variation in I, instead of obtaining result type R one obtains result type $\overline{R}$

Repeatedly small variations in initial conditions lead to repeatedly changes in result types.

Small changes - big effects

Stoss-Spiel : "small variations of the initial movement [in the Stoss-Spiel] are sufficient, to produce black instead of white" (Von Kries [14], p. 58.)

Wheel of Fortune : "Seulement, il suffit que l'impulsion [donnée à la roulette] varie d'un millième ou d'un deux-millième, pour que mon aiguille s'arrête à un secteur qui est noir ou au secteur suivant qui est rouge. " (Poincaré [5], p. 6-7.)

Roulette : "A minimal difference in launching the roulette - gain or loss of a fortune." (Smoluchowski [12], p. 255.)

Coin : "depending on whether the falling time is a little bit longer or shorter, the coin lands in the interval characterized by head or face" (Reichenbach [6] p. 50.)

Idea : Use

1. A partition whose comparison ratios can be inferred by the known symmetries of the chance set up under study
2. Comparison ratios remain invariant under linear in the small transformations :

E.g. **roulette** : Consider the net N_S on rotational angles (represented by parameter space S). If the wheel is well painted and well weighted, then black intervals are of type $]2n, 2n+1[$ red intervals are of type $]2n+1, 2n+2[$

N_S has comparison ratio $1:1$ between black and red. (If red segments are b as big as black ones, N_S has comparison ratio $1:b$.)

Let $f : \omega \to \theta$ map initial conditions, say speed, to rotational angles. If f is linear in the small (with respect to pattern size of N_θ) then the partition N_ω induced by f^- on ω has same comparison ratios.

Justifying continuity

Continuity of the distribution

1. is a **synthetic priori judgment** (Reichenbach) : it is necessary for physical knowledge and experience
2. is a **hypothesis of simplicity** (à la Poincaré) : without it mathematical representation of physical problems would become complicated if not impossible.
3. is obtained by **pure (enumerative) induction** (Strevens, von Plato, Kries)
4. is obtained from **induction base + deduction** :
   1. from the underlying mechanics (Von Plato, Hopf, Strevens)
   2. from ergodic theory (Von Plato, Hopf)

Continuity as **synthetic priori judgment**

Continuity of distribution is a necessary assumption in order to

1. have the concept of 'same magnitude in repeated trials'
2. attribute numerical values to functional physical expressions

without

1. (1) we cannot make experiments
2. (2) we could not compare empirical consequences of our hypothesis and theories, with measurement. Physical explanation, prediction and testing loose sens.

Problem : Reichenbach's argument does not provide unicity of justification.

continuity from pure (enumerative) induction

Let C be a certain comprehensible class of parameters (rotational angles of roulettes, results of chance games, 'standard variables'[1])

- We know of finitely many instances ω_n of a C that the distributions ϕ_n on ω_n are continuous.
- (by induction) for all parameters in C distributions are continuous.

Problem 1 : From the same (or less amount of) information from which we induce continuity, we could induce the concequence of (MA) (for the particular class considered).

E.g. induction for one roulette : $Card$(types of results) $\leq$ $Card$(values the parameter can take). We need more trials to make a convenient estimation about the form of the distribution on the parameter than on result types. The method becomes sterile **Problem 2** : uses the concept of 'same magnitude in repeated trials'

[1] "Variables which we use to work with" (Strevens)

Continuity from mechanics

(Plato) S is a **bridgeable** mechanical system, if S has no gap between final conditions θ of repetition n and initial conditions ω of repetition $n+1$

1. If S is periodic with respect to ω then it is for θ.
2. If period is of length n then S is in n-step auto-correlation.
3. If it is not periodic the same initial condition never obtaines twice !

ϕ results from counting the number of each condition obtained after a possibly infinite sequence of trials :

1. If S is periodic then ϕ on ω (or θ) is continuous (uniform)
2. If S is not periodic then ϕ is continuous (almoste uniform : every condition at most once)

Problem : artificial. Not independent from starting point and order on initial conditions.

Continuity from ergodic theory

Suppose S is bridgeable and **ergodic**, S has its (actually realized) initial/final conditions dense in the set of all possible conditions

$\Rightarrow \phi$ is continuous (uniform) appart from a set of measure zero.

Problem : Ergodic theory supposes absolute continuity (integrability)

(1) mechanistical (or causal) features
(2) symmetries
(3) a general probability assumption

$\Downarrow$ Explain/Infer

Unique initial probability attributions

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