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Pure Inductive Logic, Workshop Notes for Progic 2015

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A full set of notes for this course are available at
www.maths.manchester.ac.uk/~jeff/lecture-notes/Progic15.pdf

Context and Notation of Unary PIL

L_q is the first order language with

- Constant symbols a_n, $n \in \mathbb{N}^+ = \{1, 2, 3, \ldots\}$
- Predicate (i.e. unary relation) symbols $R_1, R_2, \ldots, R_q$.

$SL_q, QFSL_q$ denote the sentences and quantifier free sentences of L_q.

Let M be a structure for L_q with universe the interpretations of the a_n (also denoted a_n).

> Question: Given an agent $\mathcal{A}$ inhabiting an unknown structure M for L_q and $\theta \in SL_q$ what (subjective) probability $w(\theta)$ should $\mathcal{A}$ rationally, or logically, assign to θ?

Very Important Condition here: $\mathcal{A}$ knows nothing about M, s/he has no particular interpretation in mind for the constants and predicates.

More precisely:

> Question: Given an agent $\mathcal{A}$ inhabiting an unknown structure M for L_q, rationally or logically, what probability function w should $\mathcal{A}$ adopt?

Here $w : SL_q \to [0,1]$ is a *probability function on* L_q if for all $\theta, \phi, \exists x\, \psi(x) \in SL_q$

(P1) $\models \theta \;\Rightarrow\; w(\theta) = 1.$

(P2) $\theta \models \neg\phi \;\Rightarrow\; w(\theta \vee \phi) = w(\theta) + w(\phi).$

(P3) $w(\exists x\, \psi(x)) = \lim_{n\to\infty} w(\psi(a_1) \vee \psi(a_2) \vee \ldots \vee \psi(a_n)).$

Proposition 1 *Let w be a probability function on SL. Then for $\theta, \phi \in SL$,*

(a) $w(\neg\theta) = 1 - w(\theta).$

(b) $\models \neg\theta \;\Rightarrow\; w(\theta) = 0.$

(c) $\theta \models \phi \;\Rightarrow\; w(\theta) \leq w(\phi).$

(d) $\theta \equiv \phi \;\Rightarrow\; w(\theta) = w(\phi).$

(e) $w(\theta \vee \phi) = w(\theta) + w(\phi) - w(\theta \wedge \phi).$

The corresponding conditional probability function $w(- \,|\, -)$ is a probability function such that for $\theta, \phi \in SL_q$,

$$w(\theta \,|\, \phi) \cdot w(\phi) = w(\theta \wedge \phi), \quad \text{i.e. } w(\theta \,|\, \phi) = \frac{w(\theta \wedge \phi)}{w(\phi)} \text{ if } w(\phi) > 0.$$

Specifying Probability Functions

Gaifman's Theorem 2 *Suppose that $w : QFSL_q \to [0,1]$ satisfies (P1) and (P2) for $\theta, \phi \in QFSL_q$. Then w has a unique extension to a probability function on L_q satisfying (P1),(P2),(P3) for any $\theta, \phi, \exists x\, \psi(x) \in SL_q$.*

Example

Let $\alpha_1, \ldots, \alpha_{2^q}$, the *atoms* of L_q, denote the 2^q formulae of the form

$$R_1^{\epsilon_1}(x) \wedge R_2^{\epsilon_2}(x) \wedge \ldots \wedge R_q^{\epsilon_q}(x)$$

where the $\epsilon_i \in \{0,1\}$ and $R^1 = R, R^0 = \neg R.$

Let

$$\vec{c} \in \mathbb{D}_{2^q} = \{\langle x_1, x_2, \ldots, x_{2^q}\rangle \,|\, x_i \geq 0, \sum_{i=1}^{2^q} x_i = 1\}$$

Define $w_{\vec{c}}$ on (instantiations) of atoms by

$$w_{\vec{c}}(\alpha_j(a_i)) = c_j, \quad j = 1, 2, \ldots, 2^q,$$

Extend $w_{\vec{c}}$ to *state descriptions*, that is conjunctions of atoms, by setting, for $b_1, b_2, \ldots, b_n$ distinct elements of $\{a_k \mid k \in \mathbb{N}^+\}$,

$$\begin{aligned}
&w_{\vec{c}}(\alpha_{h_1}(b_1) \wedge \alpha_{h_2}(b_2) \wedge \ldots \wedge \alpha_{h_n}(b_n)) \\
&\qquad = w_{\vec{c}}(\alpha_{h_1}(b_1)) \times w_{\vec{c}}(\alpha_{h_2}(b_2)) \times \ldots \times w_{\vec{c}}(\alpha_{h_n}(b_n))) \\
&\qquad = c_{h_1} \times c_{h_2} \times \ldots \times c_{h_n} \\
&\qquad = \prod_{j=1}^{n} c_{h_j} = \prod_{k=1}^{2^q} c_k^{m_k} \text{ where } m_k = |\{j \mid h_j = k\}|.
\end{aligned}$$

By the Disjunctive Normal Form Theorem, for $\theta(b_1, b_2, \ldots, b_n) \in QFSL_q$

$$\theta(b_1, b_2, \ldots, b_n) \equiv \bigvee_{k=1}^{s} \bigwedge_{i=1}^{n} \alpha_{h_{ik}}(b_i)$$

for some h_{ik}.

Set

$$\begin{aligned}
w_{\vec{c}}(\theta(b_1, \ldots, b_n)) &= w_{\vec{c}}\left(\bigvee_{k=1}^{s} \bigwedge_{i=1}^{n} \alpha_{h_{ik}}(b_i)\right) \\
&= \sum_{k=1}^{s} w_{\vec{c}}\left(\bigwedge_{i=1}^{n} \alpha_{h_{ik}}(b_i)\right) \\
&= \sum_{k=1}^{s} \prod_{i=1}^{n} w_{\vec{c}}(\alpha_{h_{ik}}(b_i)) \\
&= \sum_{k=1}^{s} \prod_{i=1}^{n} c_{h_{ik}}.
\end{aligned}$$

Now $w_{\vec{c}}$ satisfies (P1-2) on $QFSL_q$.

By Gaifman's Theorem $w_{\vec{c}}$ extends to a probability function on L_q.

<u>Question</u>: Given an agent $\mathcal{A}$ inhabiting an unknown structure M for L_q, rationally or logically, what probability function w should $\mathcal{A}$ adopt?

WHAT DOES 'RATIONAL' MEAN?

Rational Principles

Current main sources:

- Symmetry
- Irrelevance
- Relevance
- Analogy

The Constant Exchangeability Principle Ex

For $\theta(a_1, a_2, \ldots, a_n) \in SL_q$ *and (distinct)* $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$

$$w(\theta(a_1, a_2, \ldots, a_n)) = w(\theta(a_{i_1}, a_{i_2}, \ldots, a_{i_n})).$$

The $w_{\vec{c}}$ satisfy Ex.

Indeed they are *exactly* the probability functions satisfying Ex and the

The Constant Irrelevance Principle

If $\theta, \phi \in QFSL$ *have no constant symbols in common then*

$$w(\theta \wedge \phi) = w(\theta) \cdot w(\phi),$$

equivalently

$$w(\theta \,|\, \phi) = w(\theta).$$

de Finetti's Representation Theorem 3 *A probability function* w *on the language* L_q *satisfies Ex just if it is a mixture of the* $w_{\vec{c}}$.

More precisely, just if

$$w = \int w_{\vec{x}} \, d\mu(\vec{x})$$

where μ *is a countably additive measure on the Borel subsets of*

$$\mathbb{D}_{2^q} = \{\langle x_1, x_2, \ldots, x_{2^q}\rangle \,|\, 0 \le x_1, x_2, \ldots, x_{2^q}, \sum_i x_i = 1\}. \tag{1}$$

Theorem 4 *Ex implies the:*

(Extended) Principle of Instantial Relevance, PIR:

For $\theta(a_1, a_2, \ldots, a_n), \phi(a_{n+1}) \in SL_q$,

$$w(\phi(a_{n+2}) \,|\, \phi(a_{n+1}) \wedge \theta(a_1, a_2, \ldots, a_n)) \ge w(\phi(a_{n+2}) \,|\, \theta(a_1, a_2, \ldots, a_n)). \tag{2}$$

Proof Let the probability function w on L satisfy Ex. Write just θ for $\theta(a_1, a_2, \ldots, a_n)$.

Then for μ the de Finetti prior for w,

$$w(\theta) = \int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\theta)\, d\mu(\vec{x}) = A \text{ say},$$

$$w(\phi(a_{n+1})\wedge\theta) = \int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\phi(a_{n+1})\wedge\theta)\, d\mu(\vec{x}) = \int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\phi(a_{n+1}))\cdot w_{\vec{x}}(\theta)\, d\mu(\vec{x}) = B \text{ say},$$

$$w(\phi(a_{n+2}) \wedge \phi(a_{n+1}) \wedge \theta) = \int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\phi(a_{n+1}))^2 \cdot w_{\vec{x}}(\theta)\, d\mu(\vec{x}),$$

and (2) amounts to

$$\left(\int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\phi(a_{n+1})) \cdot w_{\vec{x}}(\theta)\, d\mu(\vec{x})\right)^2 \leq$$

$$\left(\int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\theta)\, d\mu(\vec{x})\right) \cdot \left(\int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\phi(a_{n+1}))^2 \cdot w_{\vec{x}}(\theta)\, d\mu(\vec{x})\right), \quad (3)$$

equivalently

$$B^2 \leq A \int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\phi(a_{n+1}))^2 \cdot w_{\vec{x}}(\theta)\, d\mu(\vec{x}).$$

If $A = 0$ then this clearly holds, so assume that $A \neq 0$.

Then multiplying out

$$\begin{aligned}
0 &\leq \int_{\mathbb{D}_{2^q}} \left(w_{\vec{x}}(\phi(a_{n+1}))A - \int_{\mathbb{D}_{2^q}} w_{\vec{x}}(\phi(a_{n+1}) \cdot w_{\vec{x}}(\theta)\, d\mu(\vec{x})\right)^2 w_{\vec{x}}(\theta)\, d\mu(\vec{x}) \\
&= \int_{\mathbb{D}_{2^q}} (Aw_{\vec{x}}(\phi(a_{n+1})) - B)^2\, w_{\vec{x}}(\theta)\, d\mu(\vec{x}) \\
&= \int_{\mathbb{D}_{2^q}} \left(A^2 w_{\vec{x}}(\phi(a_{n+1}))^2 - 2ABw_{\vec{x}}(\phi(a_{n+1})) + B^2\right) w_{\vec{x}}(\theta)\, d\mu(\vec{x}) \\
&= \int_{\mathbb{D}_{2^q}} A^2 w_{\vec{x}}(\phi(a_{n+1}))^2 w_{\vec{x}}(\theta)\, d\mu(\vec{x}) - 2AB^2 + AB^2
\end{aligned}$$

and dividing by A gives (3), and the result follows. ■

Principle of Predicate Exchangeability, Px

For $\phi(R_1, R_2, \ldots, R_m) \in SL_q$, where we explicitly display the predicate symbols occurring in ϕ, and (distinct) $1 \leq i_1, i_2, \ldots, i_m \leq q$,

$$w(\phi(R_1, R_2, \ldots, R_m)) = w(\phi(R_{i_1}, R_{i_2}, \ldots, R_{i_m})).$$

The $w_{\vec{c}}$ do not satisfy Px in general.

Unary Language Invariance, ULi

A probability function w satisfies Unary Language Invariance if there is a family of probability functions w^r, one on each language L_r for $r \in \mathbb{N}^+$, such that $w = w^q$, each member of this family satisfies Px + Ex and whenever $p \leq r$ then $w^r \upharpoonright SL_p = w^p$.

The Strong Negation Principle, SN

For $\theta \in SL_q$,

$$w(\theta) = w(\theta')$$

where θ' is the result of replacing each occurrence of the relation symbol R in θ by $\neg R$.

Principles of Analogy

Is analogy a source of rational principles and if so how do they relate to existing well accepted principles like Ex + Px + SN?

In his article in the Stanford Encyclopedia of Philosophy Paul Bartha suggests a *Candidate Analogical Inference Rule* based on the slogan 'probabilistic support for future similarities is an increasing function of known past similarities'

A natural first attempt here is to take the 'similarity' to apply between constants:

The General Analogy Principle, GAP

For $\vec{a} = a_3, a_4, \ldots, a_k$ *and* $\psi(a_1, \vec{a}), \phi(a_1, \vec{a}) \in SL_q$,

$$w(\phi(a_2, \vec{a}) \mid \psi(a_1, \vec{a}) \wedge \psi(a_2, \vec{a}) \wedge \phi(a_1, \vec{a})) \geq w(\phi(a_2, \vec{a}) \mid \phi(a_1, \vec{a}))$$

GAP is *analogical support by similarity of properties of* a_1, a_2

Theorem 5 *Let* w *be a probability function on* L_q *satisfying Px + SN + Ex. Then* w *satisfies GAP just if*

$$w = c_0^{L_q} = 2^{-q}(w_{\langle 1,0,0,\ldots,0\rangle} + w_{\langle 0,1,0,\ldots,0\rangle} + \ldots + w_{\langle 0,\ldots,0,0,1\rangle}).$$

In other words

$$w(\forall x\, \alpha_i(x)) = 2^{-q} \qquad \textit{for } i = 1, 2, \ldots, 2^q,$$

so

$$w(\forall x, y \bigwedge_{i=1}^{2^q} (\alpha_i(x) \leftrightarrow \alpha_i(y))) = 1.$$

Counterpart Principle, CP

For any $\theta \in SL_q$, if $\theta' \in SL_q$ is obtained by replacing some of the predicate and constant symbols appearing in θ by (distinct) new ones not occurring in θ then

$$w(\theta \mid \theta') \geq w(\theta). \tag{4}$$

CP is *analogical support by structural similarity*

Theorem 6 *Let the probability function w on L_q satisfy ULi. Then w satisfies the Counterpart Principle, CP.*

Proof Let the ULi family consist of w^r on L_r for $r \in \mathbb{N}^+$.

Then

$$w_\infty = \bigcup_{r=1}^{\infty} w_r$$

is a probability function on the infinite (unary) language $L_\infty = \{R_1, R_2, R_3, \ldots\}$ extending w and satisfying Ex and Px.

Let θ, θ' be as in the statement of CP.

So we want to show

$$w(\theta \mid \theta') \geq w(\theta), \quad \text{i.e.} \quad w_\infty(\theta \mid \theta') \geq w_\infty(\theta).$$

We may assume that all the constant and predicate symbols appearing in θ which are common to θ' are amongst $a_1, a_2, \ldots, a_k$, $R_1, R_2, \ldots, R_g$, and that the replacements are $a_{n+i} \mapsto a_{n+i+k}$ for $i = 1, \ldots, k$ and $R_{g+j} \mapsto R_{g+j+t}$ for $j = 1, \ldots, t$.

Suppressing these common constant and predicate symbols we can write

$$\theta = \theta(a_{n+1}, a_{n+2}, \ldots, a_{n+k}, R_{g+1}, R_{g+2}, \ldots, R_{g+t}),$$
$$\theta' = \theta(a_{n+k+1}, a_{n+k+2}, \ldots, a_{n+2k}, R_{g+t+1}, R_{g+t+2}, \ldots, R_{g+2t}).$$

Let

$$\theta_{i+1} = \theta(a_{n+ik+1}, a_{n+ik+2}, \ldots, a_{n+(i+1)k}, R_{g+it+1}, R_{g+it+2}, \ldots, R_{g+(i+1)t}) \in SL_\infty$$

so $\theta_1 = \theta$, $\theta_2 = \theta'$.

Define $\tau : QFSL_1 \to SL_\infty$ by

$$\tau(R_1(a_i)) = \theta_i, \quad \tau(\neg\phi) = \neg\tau(\phi), \quad \tau(\phi \wedge \eta) = \tau(\phi) \wedge \tau(\eta), \qquad \text{etc.}$$

Define $v : QFSL_1 \to [0, 1]$ by

$$v(\phi) = w_\infty(\tau(\phi)).$$

Since w_∞ satisfies (P1-2) (on SL_∞) so does v (on $QFSL_1$).

Since w_∞ satisfies Ex + Px, for $\phi \in QFSL_1$, permuting the θ_i in $w(\tau(\phi))$ will leave this value unchanged so permuting the a_i in ϕ will leave $v(\phi)$ unchanged. i.e. v satisfies Ex.

By Gaifman's Theorem v has an extension to a probability function on L_1 which still satisfies Ex.

Hence v satisfies PIR by Theorem 4, so

$$v(R_1(a_1) \mid R_1(a_2)) \geq v(R_1(a_1)).$$

But since $\tau(R_1(a_1)) = \theta$, $\tau(R_1(a_2)) = \theta'$ this gives

$$w_\infty(\theta \mid \theta') \geq w_\infty(\theta) \quad \text{and so} \quad w(\theta \mid \theta') \geq w(\theta)$$

■

But more analogies does not necessarily mean more support, and may even have the opposite effect!

However:

Theorem 7 *Let the probability function* w *on* L_q *satisfy ULi and let*

$$\begin{aligned}\theta &= \theta(\vec{a_1}, \vec{a_2}, \vec{a_3}, \vec{R_1}, \vec{R_2}, \vec{R_3})\\ \theta' &= \theta(\vec{a_1}, \vec{a_2}, \vec{a_4}, \vec{R_1}, \vec{R_2}, \vec{R_4})\\ \theta'' &= \theta(\vec{a_1}, \vec{a_5}, \vec{a_6}, \vec{R_1}, \vec{R_5}, \vec{R_6})\end{aligned}$$

where the $\vec{a_i}, \vec{R_j}$ *are all disjoint. Then*

$$w(\theta \,|\, \theta') \geq w(\theta \,|\, \theta'').$$

Is there a 'logic of analogous reasoning by structural similarity' waiting to be discovered here?

Distance Based Support

For atoms α_i, α_j define $|\alpha_i - \alpha_j|$ to be the number of R_k on which they disagree.

E.g. for $q = 3$

$$|R_1(x) \wedge \neg R_2(x) \wedge R_3(x) - \neg R_1(x) \wedge R_2(x) \wedge R_3(x)| = 2$$

Idea now is that having 'observed' $\theta(a_1, \dots, a_m)$, seeing $\alpha_j(a_{m+1})$ provides greater or equal support for next seeing $\alpha_i(a_{m+2})$ than seeing $\alpha_k(a_{m+1})$ would if

$$|\alpha_j - \alpha_i| < |\alpha_k - \alpha_i|$$

Precisely:

The Distance Analogy Principle, DAP:

If $\theta(a_1, \dots, a_m) \in SL_q$ and $|\alpha_j - \alpha_i| < |\alpha_k - \alpha_i|$ then

$$w(\alpha_i(a_{m+2}) \,|\, \alpha_j(a_{m+1}) \wedge \theta(\vec{a})) \geq w(\alpha_i(a_{m+2}) \,|\, \alpha_k(a_{m+1}) \wedge \theta(\vec{a})).$$

Theorem 8 *For $q = 3$ the only probability functions satisfying DAP are $c^\dagger = w_{\langle 1/8, 1/8, \dots, 1/8 \rangle}$ and convex mixtures of $c_0^{L_q}$ and the probability function*

$$\begin{aligned} 12^{-1}(&w_{\langle \frac{1}{2},\frac{1}{2},0,0,0,0,0,0\rangle} + w_{\langle \frac{1}{2},0,\frac{1}{2},0,0,0,0,0\rangle} + w_{\langle \frac{1}{2},0,0,0,\frac{1}{2},0,0,0\rangle} \\ &+ w_{\langle 0,\frac{1}{2},0,\frac{1}{2},0,0,0,0\rangle} + w_{\langle 0,\frac{1}{2},0,0,0,\frac{1}{2},0,0\rangle} + w_{\langle 0,0,\frac{1}{2},\frac{1}{2},0,0,0,0\rangle} \\ &+ w_{\langle 0,0,\frac{1}{2},0,0,0,\frac{1}{2},0\rangle} + w_{\langle 0,0,0,\frac{1}{2},0,0,0,\frac{1}{2}\rangle} + w_{\langle 0,0,0,0,\frac{1}{2},\frac{1}{2},0,0\rangle} \\ &+ w_{\langle 0,0,0,0,\frac{1}{2},0,\frac{1}{2},0\rangle} + w_{\langle 0,0,0,0,0,\frac{1}{2},0,\frac{1}{2}\rangle} + w_{\langle 0,0,0,0,0,0,\frac{1}{2},\frac{1}{2}\rangle}). \end{aligned}$$

Current state of play with prospective ‘Analogy Principles’ in PIL is that they either clash with ‘must have’ principles such as Ex, Px, ULi etc. or they follow from these principles.

Since relevance principles also seem to be consequences of symmetry and irrelevance it suggests (wildly) that right now these are the only two true sources of rationality(!!)