Adams Conditionals and Non-Monotonic Probabilities

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Probabilistic Reasoning

Lord Russell's Murder

1. It is very probable that it wasn't the cook.
2. It is very probable that if it wasn't the butler, then it was the cook.

Hence:

3. It is very improbable that if it wasn't the butler, then it was the gardener.

Now:

4. It is certain, *given* that it wasn't the cook, that if it wasn't the butler, then it was the gardener.
5. It is impossible, *given* that it was the cook, that if it wasn't the butler, then it was the gardener.

Hence:

6. It is very probable that if it wasn't the butler, then it was the gardener.

But this contradicts 3.!

A Triviality Theorem

Let Ω be a set of sentences closed under the sentential operations of conjunction and negation and Π be the set of all probability measures on Ω. Let $\rightarrow$ denote a two place operation on sentences and assume that for some A and X in Ω, the sentence $A \rightarrow X$ is in Ω. Let P be a probability variable ranging over the members of Π.

(Preservation) If $P(A) > 0$ then:

(a) If $P(X) = 1$ then $P(A \rightarrow X) = 1$

(b) If $P(X) = 0$ then $P(A \rightarrow X) = 0$

Triviality Theorem: Assume the Preservation Condition. Then:

$$P(A|X) > 0,\ P(A|\neg X) > 0 \Rightarrow P(A \rightarrow X) = P(X)$$

Adams' Theses

1. Ordinary inference is governed by considerations of probabilistic validity: roughly, the high probability of the premises ensures the high probability of the conclusion.

2. The probability of a conditional is the conditional probability of its antecedent given its consequent.

Adams, E. W. (1975) *The Logic of Conditionals*, Dordrecht and Boston, Reidel.

Popper, K & D. Miller (1994) "Contributions to the Formal Theory of Probability" in *Patrick Suppes: Scientific Philosopher*, Vol 1, pp. 3 - 23, Kluwer

Probability Logic

Framework

a. Terms X, Y, Z, …, and variables, x, y, z, …, ranging over them.

b. A two-place numerical function, *p*.

c. One or more operation symbols, with the (unextended) operation of concatenation occurring in every system.

d. The domain of interpretation is left unspecified

Axiom System M

A1.	$0 \leq p(\mathrm{x}, \mathrm{z})$	(lower bound)
A2.	$p(\mathrm{x}, \mathrm{z}) \leq p(\mathrm{y}, \mathrm{y})$	(upper bound)
A3.	$\perp\mathrm{x}\perp\mathrm{z}(p(\mathrm{x}, \mathrm{z}) \neq 0)$	(non-triviality)
M1.	$p(\mathrm{xy}, \mathrm{z}) \leq p(\mathrm{x}, \mathrm{z})$	(monotonicity)
M2.	$p(\mathrm{xy}, \mathrm{z}) = p(\mathrm{x}, \mathrm{yz}).p(\mathrm{y}, \mathrm{z})$	(product rule)

In **M** it is derivable that:

1. *Idempotence:* $p(\mathrm{xx}, \mathrm{z}) = p(\mathrm{x}, \mathrm{z})$
2. *Commutativity:* $p(\mathrm{xy}, \mathrm{z}) = p(\mathrm{yx}, \mathrm{z})$
3. *Associativity:* $p(\mathrm{x}(\mathrm{yz}), \mathrm{w}) = p((\mathrm{xy})\mathrm{z}, \mathrm{w})$.

Logic

Probabilistic Implication: $x \square y =_{Df}$ ┘ $z(p(x, z) \leq p(y, z))$

Probabilistic Equivalence: $x \sim y =_{Df} x \square y$ and $y \square x$

$\square$ is a partial order and $\sim$ an equivalence relation. However, to identify probabilistically equivalent terms, replacement in *p*'s second argument must be assured:

A5. $x \sim y \Rightarrow p(z, x) = p(z, y)$

By addition of A5 to **M** we obtain system $\mathbf{M}^+$. The models of $\mathbf{M}^+$ are reducible to lower semi-lattices.

Introduction of a join operation via:

J. $p(x \vee y, z) + p(xy, z) = p(x, z) + p(y, z)$

yields axiom system $\mathbf{D}^+$ whose models are reducible to distributive lattices.

Introduction of complementation via:

C. $p(y, z) \neq p(z, z) \Rightarrow p(x, z) + p(x', z) = p(z, z)$

yields axiom system $\mathbf{B}^+$ whose models are reducible to Boolean algebras.

Conditionals

We introduce an Adams conditional as follows:

CN. $p(x \rightarrow y, z) = p(y, xz)$

The axiom system formed by adding CN to **M** we call **MCN**. In it the definition of $\square$ implies:

1. *Implication*: $x \square y$ iff ┘ $z(p(x \rightarrow y, z) = 1)$
2. *Modus Ponens*: $x(x \rightarrow y) \square y$
3. *Conditional Proof*: $zx \square y$ iff $z \square x \rightarrow y$

But … consider:

C1. $p(w, z) \neq 1 \Rightarrow p(x(x'), z) = 0$

Triviality: **MCN** $\Rightarrow p(y, z) \leq p(y, xz)$

Collapse: **MCN** $\square$ $\{C1\} \Rightarrow p$ is bivalent.

Non-Monotonic Probability

Axiom system **NM**:

A1. $0 \leq p(\mathrm{x}, \mathrm{z})$

A/2. $p(\mathrm{y}, \mathrm{y}) = 1$

A3. ┘x┘z($p(\mathrm{x}, \mathrm{z}) \neq 0$)

M/1. $p(\mathrm{y}, \mathrm{xz}) \leq p(\mathrm{x}, \mathrm{xz}) \Leftrightarrow p(\mathrm{xy}, \mathrm{z}) \leq p(\mathrm{x}, \mathrm{z})$

M2. $p(\mathrm{xy}, \mathrm{z}) = p(\mathrm{x}, \mathrm{yz}).p(\mathrm{y}, \mathrm{z})$

Theorem: The models of $\mathbf{NM}^+$ are reducible to lower semi-lattices

Example: Let F be the set all functions (partial and total) on the set S = {1,2,3} to {0,1}. We identify each f$\in$F by its domain, d(f) and the subset, v(f), of its domain that is assigned the value 1. Let:

$$\mathrm{fg(i)} = \mathrm{f(i)} \times \mathrm{g(i)}.$$

$$p(\mathrm{f}, \mathrm{g}) = \#[\mathrm{v(f)} \cap \mathrm{v(g)}] \,/\, \#[\,\mathrm{d(f)} \cap \mathrm{v(g)}]$$

Then <F, p> forms a system of non-monotonic conditional probabilities.

Probability of Conditionals (Again)

The result of adding CN to $\mathbf{NM}^+$ will be called $\mathbf{CN}^+$.

Within $\mathbf{CN}^+$:

(a) $x \to (y \to z) \sim xy \to z$

(b) $(x \to y)(x \to z) \sim x \to yz$

(c) $x(x \to y) \sim xy$

(d) $(x \to y)(xy \to z) \sim x \to yz$

(e) $x \to (y \to z) \sim (x \to y) \to (x \to z)$

(f) $x \to y \sim x \to xy$

(g) $x \square y \Rightarrow z \to x \square z \to y$

But neither Conditional Proof nor Modus Ponens are valid.

Example: Define $f \to g$ as the restriction of fg to $v(f)$

Extensions: (1) Join

By adding J to $\mathbf{NM}^+$ and $\mathbf{CN}^+$ we obtain $\mathbf{NMD}^+$ and $\mathbf{CND}^+$. Remarkably the models of $\mathbf{NMD}^+$ are reducible to distributive lattices.

Within $\mathbf{NMD}^+$:

(a) $x \vee x \,\square\, x$

(b) $x \vee y \,\square\, y \vee x$

(c) $x \vee (y \vee z) \,\square\, (x \vee y) \vee z$

(d) $x(y \vee z) \,\square\, (xy \vee xz)$

(e) $xy \vee z \,\square\, (x \vee z)(y \vee z)$

While within $\mathbf{CND}^+$:

(f) $(x \rightarrow y) \vee (x \rightarrow z) \sim x \rightarrow (y \vee z)$

Extensions: (2) Complement

C1. $p(w, z) \neq 1 \Rightarrow p(x(x'), z) = 0$

By adding C1 to **NM⁺**, **CN⁺** and **NMD⁺** we obtain **NMC⁺**, **CNC⁺** and **NMDC⁺**. The main effect of this is to ensure that the models of the resultant axiom systems are bounded by the element x(x') – hereafter designated by ⊥.

(a) Within **NMC⁺**: $y\bot \sim \bot$

(b) Within **CNC⁺**: $y \rightarrow \bot \sim \bot$

(c) Within **NMDC⁺**: $y \vee \bot \sim y$

The natural dual for C1 is the following:

C2. $p(x \vee (x'), z) = 1$

Let $\{C2\} \cup$ **NMDC⁺** be called **NMB⁺**. Not only are the models of **NMB⁺** bounded by the element $x \vee (x')$, but they are reducible to Boolean algebras.

But … CN is not non-trivially consistent with **NMB⁺**. For $\{CN\} \cup$ **NMB⁺** implies that *p* is bivalent.

Hypothesis: Logics of (Adams) conditionals cannot be both distributive and complemented.