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presenting joint work with
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Second Reasoning
Club Conference
17.06.2013—19.06.2013
Dear young reader, to understand the following story let me briefly tell you that Objective Bayesianism is a normative approach to rational belief formation stipulating that
A. Beliefs should satisfy the axioms of probability.
B. Beliefs should satisfy constraints imposed by ones evidence.
C. Beliefs should maximize entropy among the probability functions satisfying the constraints imposed by the agent’s evidence.
A Bedtime Story
Chapter 1

So spoke the all-knowing oracle: “Your beliefs shall be coherent (probabilistic). If they are not the Dutch-Book will make sure that you loose money.”
Chapter 2

So spoke the all-knowing oracle: "Your beliefs shall be calibrated. Otherwise, repeated betting will loose you money."
So spoke the all-knowing oracle: ``Your beliefs shall be maximally equivocal. Otherwise, your worst-case expectation betting returns are too low.''

*
Fin.
And since the boy was a good Objective Bayesian he slept well; every single night.
One night the son asks his dad: Why should I avoid three different types of loss (sure loss, expected loss, worst-case expected loss)?
His dad did not have an answer and our little hero had a really bad night.
Cooking up a story
Scoring Rules 101 - The usual story

- Idea: Ask DM for a forecast expressing her beliefs, i.e. \( \text{bel} : \mathcal{L} \rightarrow [0, 1] \).
- Denote by \( \Omega \) the set of states \( (\omega = \bigwedge_{1 \leq i \leq n \pm x_i}; \text{elementary events}) \).
- If \( \omega \in \Omega \) obtains, then DM will suffer loss \( L(\omega, \text{bel}) \).
- Expected loss then leads to the notion of a scoring rule
  \[
  S(P, \text{bel}) := \sum_{\omega \in \Omega} P(\omega) L(\omega, \text{bel}) .
  \]
- \( P \) is the chance function (distribution of some random variable).
- \textit{Low score is good!}
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If $\omega \in \Omega$ obtains, then DM will suffer loss $L(\omega, bel)$.

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Normally, there are other good reasons (Dutch Book, Cox’s Theorem) to adopt a probability function.

We want to give one account, which makes DM adopt a probability function, i.e. get rid of nightmares.

Thus, a scoring rule $S(P, bel)$ which only depends on the $bel(\omega)$ for $\omega \in \Omega$ is not going to cut it. – We would have no way to constrain $bel(\omega_1 \lor \omega_2)$.

Instead, we will consider extended score

$$S(P, bel) = \sum_{F \subseteq \Omega} P(F) \cdot L(F, bel)$$

compare with

$$\sum_{\omega \in \Omega} P(\omega) \cdot L(\omega, bel) .$$
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However, DM does not know $P^*$, all she knows is $P^* \in E \subseteq \mathbb{P}$. Minimizing worst case loss makes sense.

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We aim to justify adopting the $P^\dagger$ which maximizes

$$H_\Omega(P) = \sum_{\omega \in \Omega} -P(\omega) \cdot \log(P(\omega)).$$

So our loss function will have to be logarithmic.

Axioms L1 – L4 imply that $L(F, bel) = -\log(bel(F))$.

$L(F, bel) = \log(bel(F))$ is interpreted as the loss distinct to $F$, if $F$ obtains.

$$S_{\mathcal{P}\Omega}(P, B) := -\sum_{F \subseteq \Omega} P(F) \cdot \log(bel(F))$$

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Our story is along the lines: Minimize (...) logarithmic loss!

If \( \text{bel}(F) = 1 \) for all \( F \subseteq \Omega \), then \( L(F, \text{bel}) = -\log(1) = 0 \).

Thus, \( S_{\mathcal{P}\Omega}(P, \text{bel}) = \sum_{F \subseteq \Omega} P(F) \cdot 0 = 0 \).

So, \( \text{bel} \equiv 1 \) minimizes loss! This is BAD.

Houston, we have a problem!

Fact: This same problem raises its ugly head for every local extended strictly proper scoring rule.
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For $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\pi = \langle (\omega_1, \omega_2, \omega_4), (\omega_3) \rangle$ is a partition of $\Omega$.

Let $\Pi$ be the set of partitions of states of our language.

Let $M := \max_{\pi \in \Pi} \sum_{F \in \pi} \text{bel}(F)$.

Given a belief function $\text{bel} : \{F \subseteq \Omega\} \rightarrow \mathbb{R}_{\geq 0}$ ($\text{bel}$ not zero everywhere), its normalisation $B$ is defined as $B(F) := \text{bel}(F)/M$.

Set of normalized belief functions

$$
\mathbb{B} := \{B : \{F \subseteq \Omega\} \rightarrow [0, 1] : \sum_{F \in \pi} B(F) = 1 \text{ for some } \pi
$$

and

$$
\sum_{F \in \pi} B(F) \leq 1 \text{ for all } \pi \in \Pi\}.
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and $\sum_{F \in \pi} B(F) \leq 1$ for all $\pi \in \Pi$.
Normalize!

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- Given a belief function $bel : \{F \subseteq \Omega\} \rightarrow \mathbb{R}_{\geq 0}$ ($bel$ not zero everywhere), its normalisation $B$ is defined as $B(F) := \frac{bel(F)}{M}$.
- Set of normalized belief functions

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Good News Everyone!

Theorem – Norm 1, 2

For convex \( \mathcal{E} \subseteq \mathcal{P} \)

\[
\arg \inf_{B \in \mathcal{B}} \sup_{P \in \mathcal{E}} S_{P \Omega}^\log(P, B) = \arg \sup_{P \in \mathcal{E}} H_{P \Omega}(P) = \{P^\dagger_{P \Omega}\}.
\]

Theorem – Norm 1, 2, 3

If \( P^- \in \bar{\mathcal{E}} \), then

\[
\arg \inf_{B \in \mathcal{B}} \sup_{P \in \mathcal{E}} S_{P \Omega}(P, B) = \arg \sup_{P \in \mathcal{E}} H_{P \Omega}(P) = \{P^=\} = \arg \sup_{P \in \mathcal{E}} H_{\Omega}(P)
\]
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**Theorem – Norm 1, 2**

For convex $\mathcal{E} \subseteq \mathcal{P}$

$$\arg \inf_{B \in \mathcal{B}} \sup_{P \in \mathcal{E}} S_{P\Omega}^{\log}(P, B) = \arg \sup_{P \in \mathcal{E}} H_{P\Omega}(P) = \left\{ P_{\uparrow} \right\}.$$ 

**Theorem – Norm 1, 2, 3**

If $P_{\equiv} \in \bar{\mathcal{E}}$, then

$$\arg \inf_{B \in \mathcal{B}} \sup_{P \in \mathcal{E}} S_{P\Omega}(P, B) = \arg \sup_{P \in \mathcal{E}} H_{P\Omega}(P) = \left\{ P_{\equiv} \right\} = \arg \sup_{P \in \mathcal{E}} H_{\Omega}(P).$$
Not so good news

Theorem

There exists a convex $E$ such that

$$\arg \inf_{B \in B} \sup_{P \in E} S_{P \Omega}(P, B) = \{ P^\dagger_{P \Omega} \} \neq \arg \sup_{P \in E} H_{P \Omega}(P).$$
There is another plausible way to define extended score:

\[
S_{\Pi}(P, B) := \sum_{\pi \in \Pi} \sum_{F \in \pi} -P(F) \cdot \log(B(F))
\]

\[
= \sum_{F \subset \Omega} \left( \sum_{\pi \in \Pi} 1 \right) - P(F) \cdot \log(B(F))
\]

\[
H_{\Pi}(P) := S_{\Pi}(P, P).
\]
Good News Everyone!

**Theorem – Norm 1, 2**

For convex $\mathcal{E} \subseteq \mathcal{P}$

$$\arg \inf_{B \in \mathcal{B}} \sup_{P \in \mathcal{E}} S_{\Pi}(P, B) = \arg \sup_{P \in \mathcal{E}} H_{\Pi}(P) = \{ P_{\Pi}^{\dagger} \}.$$  

**Theorem – Norm 1, 2, 3**

If $P_{\perp} \in \bar{\mathcal{E}}$, then

$$\arg \inf_{B \in \mathcal{B}} \sup_{P \in \mathcal{E}} S_{\Pi}^{\log}(P, B) = \arg \sup_{P \in \mathcal{E}} H_{\Pi}(P) = \{ P_{\perp} \} = \arg \sup_{P \in \mathcal{E}} H_{\Omega}(P)$$
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**Theorem – Norm 1, 2**

For convex \( E \subseteq P \)

\[
\arg \inf_{B \in \mathcal{B}} \sup_{P \in E} S_\Pi(P, B) = \arg \sup_{P \in E} H_\Pi(P) = \{ P^\dagger_\Pi \}.
\]

**Theorem – Norm 1, 2, 3**

If \( P \perp \in \bar{E} \), then

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Theorem

There exists a convex $\mathbb{E}$ such that

$$
\arg \sup_{P \in \mathbb{E}} H_\Pi(P) \neq \arg \sup_{P \in \mathbb{E}} H_\Omega(P) \neq \arg \sup_{P \in \mathbb{E}} H_{P\Omega}(P) .
$$
There is a general way to define extended score:

\[
S_g(P, B) := \sum_{\pi \in \Pi} g(\pi) \sum_{F \in \pi} -P(F) \cdot \log(B(F))
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\[
H_g(P) := S_g(P, P).
\]

\(g : \Pi \to \mathbb{R}_{\geq 0}\) such that \(\sum_{\pi \in \Pi} g(\pi) > 0\) for all \(F \subseteq \Omega\).
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\[ g : \Pi \rightarrow \mathbb{R}_{\geq 0} \text{ such that } \sum_{\pi \in \Pi} g(\pi) > 0 \text{ for all } F \subseteq \Omega. \]
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Theorem – Norm 1, 2

For convex $\mathbb{E} \subseteq \mathbb{P}$

$$\arg \inf_{B \in \mathbb{B}} \sup_{P \in \mathbb{E}} S_g(P, B) = \arg \sup_{P \in \mathbb{E}} H_g(P) = \{ P^\dagger \} .$$

Theorem – Norm 1, 2, 3

If $P_\perp \in \overline{\mathbb{E}}$ and if $g$ is symmetric, then

$$\arg \inf_{B \in \mathbb{B}} \sup_{P \in \mathbb{E}} S_g^\log(P, B) = \arg \sup_{P \in \mathbb{E}} H_g(P) = \{ P_\perp \} = \arg \sup_{P \in \mathbb{E}} H_\Omega(P) .$$
Good News Everyone!

**Theorem – Norm 1, 2**

For convex $\mathbb{E} \subseteq \mathbb{P}$

$$\arg \inf_{B \in \mathcal{B}} \sup_{P \in \mathbb{E}} S_g(P, B) = \arg \sup_{P \in \mathbb{E}} H_g(P) = \{ P_g^+ \} .$$

**Theorem – Norm 1, 2, 3**

If $P_\equiv \in \bar{\mathbb{E}}$ and if $g$ is symmetric, then

$$\arg \inf_{B \in \mathcal{B}} \sup_{P \in \mathbb{E}} S^\log_g(P, B) = \arg \sup_{P \in \mathbb{E}} H_g(P) = \{ P_\equiv \} = \arg \sup_{P \in \mathbb{E}} H_\Omega(P)$$
Conjecture – Norm 3?

For all (reasonable) $g$ there exists a convex $\mathbb{E}$ such that

$$\arg \inf_{B \in \mathbb{B}} \sup_{P \in \mathbb{E}} H^\log_g(P) \neq \arg \sup_{P \in \mathbb{E}} H_\Omega(P).$$

Theorem – Norm 3 asterisk

For fixed $\mathbb{E}$ let $P^\dagger_g$ be the unique $g$-entropy maximizer, then

$$P^\dagger_\Omega \in \{P^\dagger_g \mid g \text{ sensible}\}.$$
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$$P^\dagger_\Omega \in \{P^\dagger_g \mid g \text{ sensible}\}.$$
The Boy sleeps well indeed - he is still very young
The loss function $L$ – Axiomatic Characterization

- **L1** $L(F, \text{bel}) = 0$, if $\text{bel}(F) = 1$.
- **L2** Loss strictly increases as $\text{bel}(F)$ decreases from 1 towards 0.
- **L3** $L$ is local. $L$ is called *local*, if and only if $L(F, \text{bel}) = L(\text{bel}(F))$.
- **L4** Losses are additive when the language is composed of independent sublanguages.

- **L1 – L4** imply that $L(\text{bel}(F)) = -\log_b(\text{bel}(F))$ for some $b \in \mathbb{R}_{>0}$. 

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- **L1 – L4** imply that $L(\text{bel}(F)) = - \log_b(\text{bel}(F))$ for some $b \in \mathbb{R}_{>0}$. 
The loss function \( L \) – Axiomatic Characterization

- **L1** \( L(F, \text{bel}) = 0 \), if \( \text{bel}(F) = 1 \).
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