

Min-Max Decision Rules for Choice under Complete Uncertainty: Axiomatic Characterizations for Preferences over Utility Intervals

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Abstract

We introduce two novel frameworks for choice under complete uncertainty. These frameworks employ intervals to represent uncertain utility attaching to outcomes. In the first framework, utility intervals arising from one act with multiple possible outcomes are aggregated via a set-based approach. In the second framework the aggregation of utility intervals employs multi-sets. On the aggregated utility intervals, we then introduce min-max decision rules and lexicographic refinements thereof. The main technical results are axiomatic characterizations of these min-max decision rules and these refinements. We also briefly touch on the independence of introduced axioms. Furthermore, we show that such characterizations give rise to novel axiomatic characterizations of the well-known min-max decision rule \succeq_{mnx} in the classical framework of choice under complete uncertainty.

Keywords: Choice under Complete Uncertainty, Interval Orders, Min-Max Decision Rules, Nonprobabilistic Decision Rules, Interval Utility
2000 MSC: 91B16, 62C20

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1. Introduction

Choice under complete uncertainty refers to a situation in which a choice has to be made among a set of known acts where the possible outcomes of each act are known, but the Decision Maker (DM) does not have any information on the (relative) probabilities of the possible outcomes (uncertainty about occurrence of events). Furthermore, the possible outcomes are totally ordered by the DM's ordinal preference relation, which allows comparisons, whether one outcome is preferred to another outcome. The final ingredient is that the DM's preferences do not allow a cardinal assessment of *how much* more an outcome is appreciated than another outcome (uncertainty about utility of outcomes), cf. [7, Chapter 3].

One important tool for the analysis of choice under complete uncertainty are investigations, in particular axiomatic characterizations, of preference relations [22]. Preference relations are here used for comparing utilities of possible outcomes, thus enabling a comparison of preferences over acts. One particular such preference relation is the *min-max decision rule* \succeq_{mnx} , introduced in [15], which represents a risk-averse DM of bounded rationality. The relation \succeq_{mnx} as well as its *lexicographic refinement* \succeq_{mnx}^L have attracted considerable interest [3, 4, 7, 9, 29].

In choice under complete uncertainty approaches the preference relations used to represent the DM's preference structure over the set of outcomes are most often linear orders. Such a representation is appropriate, for instance, if the DM can assign every possible outcome a single number reflecting an ordinal utility the outcome will yield. Staying true to the theme of *complete uncertainty* we here investigate a situation, in which such a representation is not appropriate. We will consider a DM which can only imprecisely specify the ordinal utilities attaching to outcomes.

We will here develop two frameworks where the utility obtained from a possible outcome is best represented as an interval in some connected totally ordered space X . The two frameworks we introduce differ from each other in the way they aggregate utilities from different outcomes resulting from the same act. In these frameworks we study min-max decision rules \sqsupseteq_{mnx} , \sqsupseteq_{mnx}^C , \sqsupseteq_{2mnx} , \sqsupseteq_{2mnx}^L which are natural generalizations of \succeq_{mnx} and \succeq_{mnx}^L respectively. The refinements \sqsupseteq_{mnx}^C , \sqsupseteq_{2mnx}^L enable us to distinguish between more acts, i.e., to break ties of \sqsupseteq_{mnx} respectively \sqsupseteq_{2mnx} .

The rest of the paper is organized as follows. **First** we give some examples motivating our approach, **then** we put our work into a wider context by

discussing related work. In the first technical part of the paper we introduce a set-based framework for interval-valued utilities in which the objects of choice are sets of utilities formed by finite unions of intervals, which can be visualized as utility intervals that may have gaps in them. In the [second part](#) of the paper, we introduce a vector- or multi-set-based framework in which the objects of choice are finite multi-sets of intervals whose members are considered as distinct objects even if they overlap. In both parts we shall investigate min-max decision rules and lexicographic refinements. [Finally](#) we conclude. In [Appendix A](#) and [Appendix B](#) we show how some of the axioms which we introduced for the multi-valued utility framework, can be adapted to yield an axiomatic characterization of the \succeq_{mnx} relation. Furthermore, we improve upon Arlegi’s axiomatic characterization of \succeq_{mnx} given in [4].

2. Motivating Examples

Example 1: At a fair you have the opportunity to buy a ticket for one of three raffles, each of which is a lottery that offers a chance of winning one out of a number of plush toys. The toys that may be won in each raffle are on display, but the chances of winning a given toy in a given raffle are not exactly known. You plan to take the toy home and give it to your nephew, if you did purchase a winning ticket. So the utility you obtain from a winning a certain toy (possible outcome) depends on the appreciation of a third person (your nephew).

Example 2: Consider a student deciding when to e-mail in a summer break take-home assignment. If the student sends the assignment off right away, then the resulting grade may either be a “B” or a “C”. If the student works two further weeks on it and then sends it off, the grade may possibly improve to an “A”. Given that a better grade is likely to lead to better (paying) job offers she prefers a better grade to a worse grade.

Example 3: A student in the process of graduating from university with a business degree is looking for a permanent job. Several companies in different cities hold assessment tests on the same day. A company typically has several different vacancies in different branches at the same time. The salary for such a job may contain a variable part, which depends on the performance of the employee as well as the whole company. Depending on the student’s performance she is invited to interview for a subset of the available positions.

The common theme of these examples is that a DM may be reluctant to assign a single number to each outcome representing her utility. More-

over, the DM might not be willing to specify a single probability function expressing her beliefs in obtaining a certain utility from an outcome. The DM may feel more comfortable assigning each outcome a range of utilities it may possibly yield without making any assertion concerning the likelihood of the outcomes yielding these utilities. Thus, the DM's preferences over acts (with in general more than one possible outcome) are best be represented by preferences over sets of intervals of utility.

3. Related Work

3.1. Choice under Complete Uncertainty

Choice under complete uncertainty can be understood as subfield of *ranking sets of objects* [6, 7, 21, 27, 34, 42]. The latter is the study of how to extend preferences defined over objects to preferences over sets of such objects. The former interprets these objects as possible outcomes and acts as sets of possible outcomes. A DM's relative preference for one act over another can thus be understood as relative preference of one set of possible outcomes over another such set. A good albeit slightly outdated overview of this field can be found in [7, Section 3].

Bossert & Slinko [16] studied uncertainty aversion in set-based models of choice under complete uncertainty. They provided a complete ranking of certain decision rules according to relative degree of uncertainty aversion. Gravel et al. [29] took a slightly different approach, characterizing decision rules under complete uncertainty via expected utility. Arlegi [3] showed how some well-known decision rules under complete uncertainty can be reinterpreted in procedural terms. From such a perspective the DM evaluates possible acts by comparing certain *focal* elements in the set of possible outcomes of these acts, losing sight or even completely ignoring all non-focal elements. Decision models using only a limited amount of information are of particular interest in psychology [28]. Ben Larbi et al. investigated strategies of agents playing *multiple outcomes games* [8]. They thus demonstrated the relevance of choice under complete uncertainty to game theory.

Recall that in the framework of choice under complete uncertainty it is assumed that the DM knows all possible outcomes resulting from each act. If we instead assume that the DM cannot envision the possible outcomes of acts, then choices are said to be taken under *complete ignorance*. Ben Larbi et al. [9] axiomatically investigated a min-max decision rule under complete ignorance. They then compare their results to those of [4] and [15].

3.2. Uncertain Utility

Decision making with acts of known but uncertain outcomes, in which the DM cannot specify the subjective expected utility of an outcome, have featured prominently in the literature. We concentrate here only on the aspects of such investigations that are more relevant to us, see [11, 12] for classical overviews. Building on the work of Kreps [38], Olszewski and Ahn studied choices between sets of lotteries in [1, 40]. Ahn [1] presented a theory of objective ambiguity without a state space. Olszewski [40] considered a framework where Nature chooses a particular lottery from the selected set of lotteries. Vierø [44] considered acts which map to sets of lotteries and gave an axiomatic characterization of certain decision rules. Jaffray & Jeleva [32] investigated acts that are only partially analyzable.

A further approach describes uncertain utility by interval probabilities, for which the axiomatic foundations were laid by Weichselberger [47]. Kozine & Utkin [37] used these probabilities to study Markov chains. For a recent overview of the many applications of interval probabilities refer to Augustin & Coolen [5, Section 2]. Range based utilities, with or without interval probabilities [46], have found their way into more practical applications. Such ranges have successfully been applied in (group) decision making and recommender systems, e.g. in [20, 30, 33, 35, 36, 43] and in the economics literature, exemplary we mention [2, 18].

3.3. Interval Orders

The problem of ordering intervals is well-known and well-studied. The most important notion in this field is that of an *interval order*, cf. [23, 25]. An interval order I_O on an ordered space $(X, >)$ orders subsets of X such that for all $Y, Z \subset X$ and $Y I_O Z$ it holds that the minimum of Y is greater than the maximum of Z , where the minimum and maximum are according to $>$. However, the focus of this research field is on the representability of interval orders by real-valued functions [13, 17, 19, 39] and not on axiomatic characterizations of preference relations as it is here.

Preference relations over intervals taking only endpoints into account are by design a rather simple class. Thus, preference orders over intervals taking also selected interior points into considerations have been developed. In [41] a general framework is developed for the comparisons of “n-point intervals”. However, this framework only allows for comparisons of intervals having the same number of points.

4. The Min-Max Relation for Intervals

4.1. The Formal Framework

We now introduce our *set-based* approach; for discussions on the merits of the set-based approach see [7, 15, 42]. We let \mathcal{A} be the set of acts, O be the set of possible outcomes and U be the possible utilities obtained from the $o \in O$. We here follow approaches in “choice under complete uncertainty” in which utilities are measured on an ordinal scale. The notion of “utility” used here is thus different from the classical notion of utility introduced by von Neumann and Morgenstern. For a detailed introduction to choice under complete uncertainty motivating the set-up in detail we refer the reader to [7, Section 3].

Denote by O_a the set of possible outcomes for act $a \in A$, which we assume to be finite. A set O_a containing more than one element is interpreted as an uncertain prospect where the DM does not have any information on the likelihood over the possible outcomes in O_a . Now assume for a moment that the utility obtained from every possible outcome $o \in O$ can be described by a single value $u \in U$. Furthermore, consider an act $a \in \mathcal{A}$ with multiple possible outcomes, where two of these outcomes, $o, o' \in O_a$ say, yield the same utility $u \in U$. The DM thus knows that obtaining utility u from act a is possible. The presence of multiple outcomes yielding the same utility does not give the DM any further information concerning the question: “Which values $u \in U$ may an act $a \in A$ possibly yield?”

As it is customary in decision science, for the purpose of deciding how to act, we identify an act $a \in \mathcal{A}$ with the utility/utilities it may yield. Thus, if φ maps every $o \in O$ to the utility DM obtains from it, we can represent an act $a \in \mathcal{A}$ by the set of utilities $\bigcup_{o \in O_a} \varphi(o) \subseteq U$.

For a DM possessing information on the relative likelihoods of outcomes for a given act, the question of which outcome may yield which utility would surely be relevant. However, we here have a situation where this is not the case. Plausibly, we can thus represent an act in such a situation by the $u \in U$ it may possibly yield.

Let us now backtrack and assume that the utility obtained from a possible outcome $o \in O$ consists possibly of more than one $u \in U$. Following the reasoning above, we identify an act $a \in \mathcal{A}$ with $\bigcup_{o \in O_a} \varphi(o) \subseteq U$ where now the $\varphi(o)$ may contain more than value. If O_a contains only one single element, then the act a does not have any uncertain outcomes. The utility obtained

from such an act is deterministic, if and only if for the unique $o_a \in O_a$ it holds that $\varphi(o_a)$ consists of a unique $u \in U$.

Bossert et al. introduced \succeq_{mnx} in [15] where they considered a finite set of outcomes. There, the DM was assumed to have a reflexive, transitive and complete binary preferences on O , thus the DM's preferences are given by an injective function $f : O \rightarrow \mathbb{N}$. In their framework O and U can thus be understood as finite subsets of \mathbb{N} .

We here identify U with \mathbb{R} and assume that the utility obtained from an $o \in O_a$ is represented by a compact interval in \mathbb{R} .² The complete linear order on \mathbb{R} will simply be denoted by $>$. Thus, for all $a \in \mathcal{A}$ the set $\bigcup_{o \in O_a} \varphi(o) \subset \mathbb{R}$ consists of finitely many connected components of \mathbb{R} . Hence, acts $a \in \mathcal{A}$ can be evaluated by comparing finite unions of compact intervals. For example, if $O_a = \{o, o', o''\}$ with $\varphi(o) = [1.5, 1.7]$, $\varphi(o') = [1.67, 2]$, $\varphi(o'') = [1, 1.3]$, then $\bigcup_{o \in O_a} \varphi(o) = [1, 1.3] \cup [1.5, 2]$.

²Let us be absolutely clear here. We could in general assume that U is some infinite connected topological space with a complete linear order. Nothing hinges on the particular space, we will here simply use \mathbb{R} . Since the preference relations we consider are invariant under order preserving transformations of the underlying space; here \mathbb{R} ; the canonical structure on \mathbb{R} only carries meaning here in as far as it allows *ordinal* comparisons.

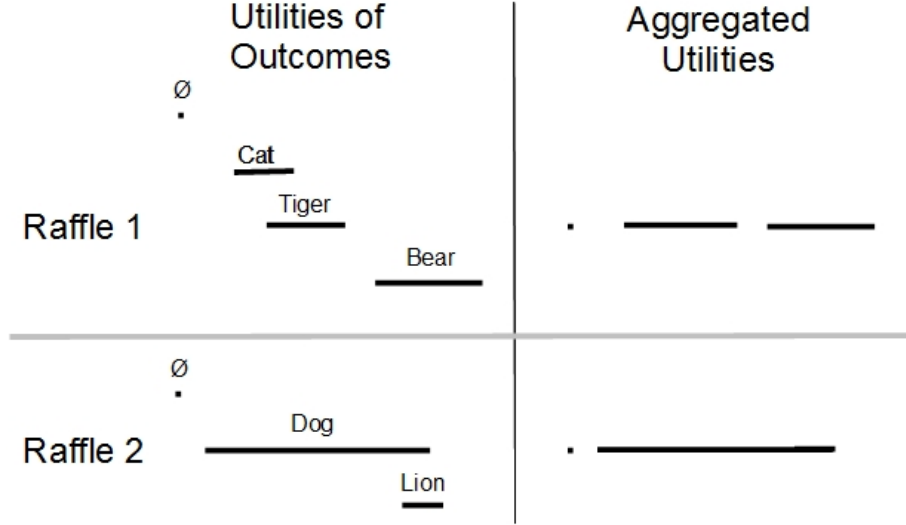


Figure 1: On the left: Two raffles as described in Example 1 with possible outcomes and uncertain utilities attaching to the outcomes. On the right: Uncertain utilities attaching to outcomes are aggregated via φ .

4.2. Intervals

A set of the form $[s, t] := \{r \in \mathbb{R} | s \leq r \leq t\}$ is an *interval* in \mathbb{R} . The degenerate interval $[s, s]$, which consists of a single number, will be denoted merely by $[s]$ to simplify notation. We will not consider the empty set to be an interval. Let INT be the set of intervals and let \mathcal{I} be the set of all finite unions of intervals, i.e.

$$\mathcal{I} := \left\{ \bigcup_{f \in F} I_f \mid I_f \in INT \text{ for all } f \in F, F \text{ is finite} \right\} . \quad (1)$$

We define the *size* of a $J \in \mathcal{I}$ as the number of connected components it possesses as a subset of \mathbb{R} , which will be denoted by $\#J$. For $J \in \mathcal{I}$ let $\underline{j}, \bar{j} \in \mathbb{R}$ denote, respectively, the minimum and maximum of J with respect to the standard order $>$ on \mathbb{R} .

Let \sqsupseteq be an ordering over \mathcal{I} , i.e. it is a reflexive, transitive and complete binary relation. This ordering is interpreted as the DM's preference structure over the uncertain outcomes, which we want to investigate. Let \sqsubset and \approx denote, respectively, the asymmetric and symmetric parts of \sqsupseteq .

Reflexivity and transitivity of preferences are such widely - though by no means universally - accepted assumptions in decision science that we shall

not dwell on them here. The completeness assumption is more contentious. Note that all we will assume here is that whenever the DM is presented with two acts and their possible outcomes, the DM can decide whether the former act is at least of equal preference as the latter act. The DM is allowed to conclude that both acts are of equal preference. For example, the simple decision rule to avoid worst cases gives rise to a preference order P with a plethora of ties. Note that we do not require the DM to be able to efficiently communicate the complete preference relation P nor do we require that P is fully open to introspection. In light of these comments the assumption of completeness appears plausible.

4.3. Axioms and Preference Relations for Intervals

We now introduce a first set of axioms. With the exception of the substitution axiom, all other axioms have already appeared in the standard choice under uncertainty framework (in non-interval form) in [4] and [15].

Interval Simple Monotonicity (ISM): For all $r, s \in \mathbb{R}$ such that $r > s$ we have $[r] \sqsupseteq [s, r] \sqsupseteq [s]$.³

Interval Simple Dominance 1 (ISD1): For all $r, s, t \in \mathbb{R}$ such that $r > s > t$ we have $[t, r] \sqsupseteq [t, s]$.

Interval Simple Dominance 2 (ISD2): For all $r, s, t \in \mathbb{R}$ such that $r > s > t$ we have $[s, r] \sqsupseteq [t, r]$.

Interval Simple Uncertainty Aversion (ISUA): For all $r, s, t \in \mathbb{R}$ such that $r > s > t$ we have $[s] \sqsupseteq [t, r]$.

Interval Simple Uncertainty Appeal (ISUP): For all $r, s, t \in \mathbb{R}$ such that $r > s > t$ we have $[t, r] \sqsupseteq [s]$.

Interval Weak Substitution (IWSUB): For all $I, K \in INT$ with $\bar{i} < \underline{k}$ we have $[\bar{i}, \bar{k}] = [\bar{i}, \underline{k}] \cup K \sqsupseteq I \cup K \sqsupseteq I \cup [\bar{i}, \underline{k}] = [\underline{i}, \underline{k}]$.

Interval Substitution (ISUB): For all $J \in \mathcal{I}$ with $J = J_1 \cup J_2 \cup \dots \cup J_{\#J}$ and for all $r \geq s \geq \bar{j}_l \in \mathbb{R}$ with $J \cap \text{int}([s, r]) = \emptyset$ we have $(J \setminus J_l) \cup [s, r] \sqsupseteq J$ and for all $\underline{j}_l \geq r \geq s \in \mathbb{R}$ with $J \cap \text{int}([s, r]) = \emptyset$ we have $J \sqsupseteq (J \setminus J_l) \cup [s, r]$.

Interval Monotone Consistency (IMC): For all $H, J \in \mathcal{I}$ with $H \sqsupseteq J$ we have $H \cup J \sqsupseteq J$.

Interval Robustness (IROB): For all $H, J \in \mathcal{I}$ with $H \sqsupseteq J$ we have $H \sqsupseteq H \cup J$.

³This standard abuse of notation, formally correct and cumbersome notation would be: “[r] \sqsupseteq [s, r] and [s, r] \sqsupseteq [s]”. We will continue to abuse the notation in this way.

The dominance and monotonicity axioms formalize the idea that better outcomes are strongly preferred. ISUA and ISUP fix the DM's attitude towards uncertain utilities.

In ISUB *int* denotes the *interior* of the interval $[s, r] \subset \mathbb{R}$ in the standard topology on \mathbb{R} , i.e. $int([s, r]) = \{t \in \mathbb{R} | s < t < r\}$. If $s = r$, then the interior of the interval $[s, r]$ is empty. Roughly speaking, the axioms says that removing some connected component $[s_1, r_1]$ from J and replacing it by an interval some other interval $[s_2, r_2]$, which is not already in J and such that $r_1 < s_2$, yields a union of intervals J' which is strictly preferred to J or of equal preference to J .

Note that ISUB is in general stronger than IWSUB. Furthermore, IWSUB implies for $i < k$ that $[i, k] \approx [i] \cup [k]$.

We now define the min-max relation (\sqsupseteq_{mnx}) and the max-min relation ($\sqsupseteq_{m xn}$) on $\mathcal{I} \times \mathcal{I}$. The strict parts are defined on pairs $H, J \in \mathcal{I}$ as follows

$$H \sqsupseteq_{mnx} J, \text{ if and only if } [\underline{h} > \underline{j} \text{ or } (\underline{h} = \underline{j} \text{ and } \bar{h} > \bar{j})] \quad (2)$$

$$H \sqsupseteq_{m xn} J, \text{ if and only if } [\bar{h} > \bar{j} \text{ or } (\bar{h} = \bar{j} \text{ and } \underline{h} > \underline{j})] . \quad (3)$$

Under this min-max relation two acts are compared primarily by their worst possible outcome. If the worst possible outcome of an act is preferred to the worst possible outcome of another act, then the former act is preferred. If the worst possible outcomes are of equal preference, then the DM does not necessarily consider both acts to be of equal preference. In such cases, ties are broken by comparing best possible outcomes. Only in the case of equal preference of worst possible outcomes and equal preference of best possible outcomes is there indifference between acts. For example in the situation depicted in Figure 1 a DM applying the min-max rule prefers a ticket from raffle 1 over a ticket from raffle 2. For both raffles the worst possible outcome is of the same utility while the best outcome for raffle 1 is preferable to the best outcome for raffle 2.

Clearly, \sqsupseteq_{mnx} and $\sqsupseteq_{m xn}$ are in a natural sense dual to each other in that max and min have swapped roles. So every true statement about \sqsupseteq_{mnx} can be turned into true statement about $\sqsupseteq_{m xn}$ by an appropriate dualization and vice versa. For the remainder we will concentrate on \sqsupseteq_{mnx} . We shall use \approx_{mnx} to denote the symmetric part of \sqsupseteq_{mnx} ; so if $H \sqsupseteq_{mnx} J$ and $J \sqsupseteq_{mnx} H$, then $H \approx_{mnx} J$.

\sqsupseteq_{mnx} is uncertainty-averse, $\sqsupseteq_{m xn}$ is uncertainty-seeking. For a discussion of uncertainty-aversion and uncertainty-seeking in nonprobabilistic decision

models refer to [14].

4.4. A First Axiomatic Characterisation of the Min-Max Relation

Let us consider for a moment the set of intervals INT and the preference relation \sqsubseteq_{mnx} restricted to INT . For two different intervals, $[s_1, r_1]$ and $[s_2, r_2]$ say, at least one of the following inequalities holds: $s_1 > s_2$, $s_2 > s_1$, $r_1 > r_2$, $r_2 > r_1$. For example, if $s_1 > s_2$, then $[s_1, r_1] \sqsubseteq_{mnx} [s_2, r_2]$. In general, it follows that either $[s_1, r_1] \sqsubseteq_{mnx} [s_2, r_2]$ or $[s_2, r_2] \sqsubseteq_{mnx} [s_1, r_1]$. Thus, \sqsubseteq_{mnx} is complete on INT and there are no ties between different intervals.

We shall now see that the uncertainty aversion axiom together with the monotonicity axiom and the first dominance axiom are jointly strong enough to ensure that every preference relation in our sense (reflexive, transitive and complete) agrees with the min-max rule when comparing pairs $M, N \in INT$.

Lemma 1. \sqsubseteq satisfies ISM, ISD1 and ISUA, if and only if \sqsubseteq agrees with \sqsubseteq_{mnx} on INT .

\sqsubseteq satisfies ISM, ISD2 and ISUP, if and only if \sqsubseteq agrees with \sqsubseteq_{mxn} on INT .

Proof If \sqsubseteq agrees with \sqsubseteq_{mnx} on INT , then it clearly satisfies ISM, ISD1 and ISUA.

Conversely, assume that \sqsubseteq satisfies ISM, ISD1 and ISUA. It is sufficient to prove that for all $M, N \in INT$

$$(M \approx_{mnx} N \text{ implies } M \approx N) \text{ and } (M \sqsubseteq_{mnx} N \text{ implies } M \sqsubseteq N) . \quad (4)$$

If intervals M, N are such that $M \approx_{mnx} N$, then $M = N$ and by the reflexivity of \sqsubseteq we find $M \approx N$.

Next consider intervals M, N such that $M \sqsubseteq_{mnx} N$. There are four cases to consider: (i) $\underline{m} = \overline{m}$ and $\underline{n} = \overline{n}$, (ii) $\underline{m} < \overline{m}$ and $\underline{n} = \overline{n}$, (iii) $\underline{m} = \overline{m}$ and $\underline{n} < \overline{n}$ and (iv) $\underline{m} < \overline{m}$ and $\underline{n} < \overline{n}$.

If (i) holds, then $\overline{m} = \underline{m} > \underline{n} = \overline{n}$. By ISM we have $M \sqsubseteq N$.

If (ii) holds, then either $\overline{m} > \underline{m} = \underline{n} = \overline{n}$ or $\overline{m} > \underline{m} > \underline{n} = \overline{n}$. By ISM we have $[\underline{m}, \overline{m}] \sqsubseteq [\underline{m}]$ and in the former case of (ii) we also have $[\underline{m}] \approx [\underline{n}] = N$. Transitivity of \sqsubseteq now implies $M \sqsubseteq N$. In the latter case of (ii) we have $[\underline{m}, \overline{m}] \sqsubseteq [\underline{m}] \sqsubseteq [\underline{n}] = N$ by ISM. Again using the transitivity of \sqsubseteq gives $M \sqsubseteq N$.

If (iii) holds, then $\overline{m} = \underline{m} > \underline{n}$. Since $\underline{n} < \overline{n}$ we either have $\underline{m} = \overline{m} \geq \overline{n} > \underline{n}$ or $\overline{n} > \underline{m} = \overline{m} > \underline{n}$. In the former case ISM and transitivity imply $[\underline{m}] \sqsubseteq$

$[\bar{n}] \sqsupseteq [\underline{n}, \bar{n}]$. In the latter case an application of ISUA yields $M \sqsupseteq N$.
 If (iv) and $M \sqsupseteq_{mnx} N$ hold, then one of the following four conditions has to hold:

$$\bar{m} > \underline{m} > \bar{n} > \underline{n}, \quad (5)$$

$$\bar{m} > \underline{m} = \bar{n} > \underline{n}, \quad (6)$$

$$\bar{m} > \underline{m} \text{ and } \bar{n} > \underline{m} > \underline{n}, \quad (7)$$

$$\bar{m} > \bar{n} > \underline{m} = \underline{n}. \quad (8)$$

If (5) holds, then by ISM and transitivity of \sqsupseteq we find $[\underline{m}, \bar{m}] \sqsupseteq [\underline{m}] \sqsupseteq [\bar{n}] \sqsupseteq [\underline{n}, \bar{n}]$.

If (6) holds, then by ISM and transitivity of \sqsupseteq we obtain $[\underline{m}, \bar{m}] \sqsupseteq [\underline{m}] \approx [\bar{n}] \sqsupseteq [\underline{n}, \bar{n}]$.

If (7) holds, then ISM, ISUA and transitivity of \sqsupseteq we have $[\underline{m}, \bar{m}] \sqsupseteq [\underline{m}] \sqsupseteq [\underline{n}, \bar{n}]$.

In case of (8) we apply ISD1 to obtain $M \sqsupseteq N$.

The second equivalence will not be proved here. The proof is either along the lines used above or the above mentioned dualization argument may be applied. \blacksquare

Theorem 2. *If \sqsupseteq satisfies ISUB, IMC and IROB, then $J \approx [j, \bar{j}]$ for all $J \in \mathcal{I}$.*

In plain English this means that any holes in J are ignored by \sqsupseteq .

Proof For $J = [j, \bar{j}]$ this is self-evident.

Let us now assume that J contains some holes, i.e. $J = \bigcup_{i=1}^{\#J} [r_i, s_i]$ with $r_i \leq s_i$ for all $1 \leq i \leq \#J \geq 2$ and $s_i < r_{i+1}$ for all $1 \leq i \leq \#J - 1$.⁴ We have by ISUB that $J \sqsupseteq [s_1, r_2] \cup \bigcup_{i=1}^{\#J-1} [r_i, s_i]$. Since \sqsupseteq is reflexive, i.e. $J \sqsupseteq J$, we can now apply IROB to obtain $J \sqsupseteq [r_1, s_2] \cup \bigcup_{i=3}^{\#J} [r_i, s_i]$. Repeating this procedure eventually yields $J \sqsupseteq [j, \bar{j}]$.

Conversely we have by ISUB $[s_1, r_2] \cup \bigcup_{i=2}^{\#J} [r_i, s_i] \sqsupseteq J$. Reflexivity of \sqsupseteq and IMC imply $[r_1, s_2] \cup \bigcup_{i=3}^{\#J} [r_i, s_i] \sqsupseteq J$. So we have managed to fill the gap between s_1 and r_2 . We now proceed as often as necessary ($\#J - 2$ times to be exact) to fill the remaining gaps. We eventually obtain $[r_1, s_{\#J}] = [j, \bar{j}] \sqsupseteq J$.

Noting that $J \sqsupseteq [j, \bar{j}] \sqsupseteq J$ implies $J \approx [j, \bar{j}]$ completes the proof. \blacksquare

⁴Note that the inductive arguments we give in this proof require J to be a *finite* union.

From the above lemma it follows that in the presence of ISM, ISD1 and ISUA we can replace IWSUB equivalently with Interval Weak Substitution' (IWSUB'): For all $I, K \in INT$ with $[k] \sqsupseteq [\bar{i}]$ we have $[\bar{i}, \bar{k}] = [\bar{i}, \underline{k}] \cup K \sqsupseteq I \cup K \sqsupseteq I \cup [\bar{i}, \underline{k}] = [\underline{i}, \underline{k}]$.

Hence, in the following axiomatic characterization of \sqsupseteq_{mnx} in Corollary 3, the only axioms mentioning the order $>$ are ISM, ISD1 and ISUA.

Corollary 3.

\sqsupseteq satisfies IWSUB', IMC, IROB, ISM, ISD1 and ISUA, if and only if $\sqsupseteq = \sqsupseteq_{mnx}$.

\sqsupseteq satisfies IWSUB', IMC, IROB, ISM, ISD2 and ISUP, if and only if $\sqsupseteq = \sqsupseteq_{mnx}$.

Proof It is straightforward to check that \sqsupseteq_{mnx} satisfies these six axioms.

Conversely, we may assume that \sqsupseteq satisfies ISM, ISD1 and ISUA. It follows from Lemma 1 that \sqsupseteq agrees with \sqsupseteq_{mnx} on INT . It remains to show that \sqsupseteq agrees with \sqsupseteq_{mnx} on all of \mathcal{I} .

Now let $J \in \mathcal{I}$, we will show that $J \approx [j, \bar{j}]$. We shall show this by induction on $\#J$. If $\#J = 1$, then $J = [j, \bar{j}]$ and hence $J \approx [j, \bar{j}]$.

If $\#J = 2$, then there are $I, K \in INT$ with $J = I \cup K$ and $\bar{i} < \underline{k}$. By IWSUB' we have $[\bar{i}, \bar{k}] \sqsupseteq J$. Since also $J \approx J$ we find via IMC that $[j, \bar{j}] = J \cup [\bar{i}, \bar{k}] \sqsupseteq J$. We also have by IWSUB' that $J \sqsupseteq [\underline{i}, \underline{k}]$. Applying IROB yields $J \sqsupseteq J \cup [\underline{i}, \underline{k}] = [j, \bar{j}]$. Hence, $[j, \bar{j}] \sqsupseteq J \sqsupseteq [j, \bar{j}]$.

If $\#J = 3$, then $J = I \cup K \cup M$ with $I, K, M \in INT$ and $\bar{i} < \underline{k} \leq \bar{k} < \underline{m}$. From the above case we already know that $[\underline{i}, \bar{m}] \approx I \cup M$. On the other hand $[\underline{i}, \bar{m}] \sqsupseteq [\underline{i}, \bar{k}] \approx I \cup K$. Applying IROB we find $[\underline{i}, \bar{m}] \approx I \cup M \sqsupseteq (I \cup K) \cup (I \cup M) = I \cup K \cup M = J$. Furthermore, we know that $I \cup M \approx [\underline{i}, \bar{m}]$ and $K \cup M \approx [\underline{k}, \bar{m}] \sqsupseteq [\underline{i}, \bar{m}] \approx I \cup M$. Applying IMC gives $J = I \cup K \cup M \sqsupseteq I \cup M \approx [\underline{i}, \bar{m}]$. Hence, we have shown that $J \approx [j, \bar{j}]$ for $\#J \leq 3$.

If $\#J > 3$, then $J = \bigcup_{i=1}^{\#J} [r_i, s_i]$ with $r_i \leq s_i$ for all $1 \leq i \leq \#J$ and $s_i < r_{i+1}$ for all $1 \leq i \leq \#J - 1$. Then $[j, \bar{j}] \approx J \setminus [r_2, s_2] \approx J \setminus [r_3, s_3]$ by the induction hypothesis. Applying IROB and IMC yields $J \approx [j, \bar{j}]$.

The part of the proof concerning \sqsupseteq_{mnx} is omitted. ■

4.5. Independence of Axioms

Lemma 4. ISUA is independent of ISM, ISD1, ISD2, IWSUB, IROB and IMC.

Proof $\sqsupseteq_{m \times n}$ satisfies ISM, ISD1, ISD2, IWSUB, IMC, IROB and ISUP. $\sqsupseteq_{m \times n}$ does not satisfy ISUA. ■

Lemma 5. *ISM is independent of ISD1, ISUB, ISUA, IROB and IMC.*

Proof Consider the preference relation \sqsupseteq_{min} defined for $H, J \in \mathcal{I}$ by $H \sqsupseteq_{min} J$ if and only if $\underline{h} \geq \underline{j}$. This relations fails to satisfy ISM but does satisfy the other five axioms. ■

Lemma 6. *IWSUB is independent of ISM, ISD1, ISD2, ISUA, IROB and IMC.*

Proof We define a preference relation \sqsupseteq^W that differs from $\sqsupseteq_{m \times n}$ by breaking certain ties. Consider $J, J' \in \mathcal{I}$ such that $J \approx_{m \times n} J'$. If J is of the form $J = [\underline{j}, k] \cup [\bar{j}, \bar{j}]$ with $\underline{j} < k < \bar{j}$, then $J' \sqsupseteq^W J$, if J' is not of this form. Since IWSUB implies for $t < s < r$ that $[t, s] \cup [r] \sqsupseteq [t, r]$ this relation does not satisfy IWSUB. ■

This last result shows, that ISM, ISD1, ISUA, IROB and IMC are not strong enough to characterize $\sqsupseteq_{m \times n}$. This is in contrast to the characterization of $\succeq_{m \times n}$ in [4] which only requires SM, SD1, SUA, ROB and MC to hold.

4.6. Further Characterizations of the Min-Max Relation

Arlegi's proof in [4] contains an axiomatic characterization of the min-max relation that corrected an erroneous proof in [15]. To obtain this characterization three new axioms are required. It is however possible to achieve this characterization without introducing any new axioms. We merely have to slightly modify one of the axioms introduced in [15], see Theorem 24 on page 31 for details. Interestingly, we can do a very similar proof in our interval framework. The axiom we need to introduce is a close relative of the Axiom of Independence. The prototype of this axiom was first introduced in [26] and later fruitfully applied, for instance in [34].

Interval Independence (IIND): For all $H, J \in \mathcal{I}$ and all $I \in INT$ with $I \sqsupseteq H \sqsupseteq J$ we have $H \cup I \sqsupseteq J \cup I$.

Theorem 7. *If \sqsupseteq satisfies IIND, ISM, ISD1, IWSUB and ISUA, then for all $J \in \mathcal{I}$ we have $J \approx [j, \bar{j}]$.*

Proof The proof is done by induction $\#J$. For $\#J = 1$, there is nothing to prove.

We shall show in Lemma 12 on page 19 that ISD2 follows from ISM and ISUA. We may thus apply this axiom here.

If $\#J = 2$, then let $J = [r, s] \cup [u, v]$ with $v \geq u > s \geq r$. By ISM we have $[u, v] \sqsupseteq [r, s]$ and hence by IIND we obtain $[u, v] \sqsupseteq [r, s] \cup [u, v] = J$. By IWSUB we have $J \sqsupseteq [r, u]$. Applying IIND to $J \sqsupseteq [r, u]$ yields $J \cup [u, v] = J \sqsupseteq [r, u] \cup [u, v] = [\underline{j}, \bar{j}]$. Now for the other direction we have by ISD1 $[r, v] \sqsupseteq [r, s]$ and by ISD2 we obtain $[u, v] \sqsupseteq [r, v]$. Applying IIND yields $[\underline{j}, \bar{j}] = [r, v] = [r, v] \cup [u, v] \sqsupseteq [r, s] \cup [u, v] = J$.

For $\#J \geq 3$ we let $J = I_1 \cup \dots \cup I_{\#J}$ where $\underline{i}_l \leq \bar{i}_l < \underline{i}_{l+1}$ and $I_l \in INT$ for all l . By the induction hypothesis and ISM we find

$$I_1 \cup \dots \cup I_{\#J-1} \approx [\underline{j}, \bar{i}_{\#J-1}] \text{ and } I_{\#J} \sqsupseteq [\underline{j}, \bar{i}_{\#J-1}] . \quad (9)$$

Applying IIND yields $[\underline{j}, \bar{i}_{\#J-1}] \cup I_{\#J} \sqsupseteq I_1 \cup \dots \cup I_{\#J} = J$. Applying the induction hypothesis to $[\underline{j}, \bar{i}_{\#J-1}] \cup I_{\#J}$ yields

$$[\underline{j}, \bar{j}] \approx [\underline{j}, \bar{i}_{\#J-1}] \cup I_{\#J} \sqsupseteq I_1 \cup \dots \cup I_{\#J} = J .$$

Addressing the other direction note that by ISM and the induction hypothesis we have $I_{\#J} \sqsupseteq [\underline{i}_{\#J}] \sqsupseteq [\bar{i}_{\#J-1}] \sqsupseteq [\underline{j}, \bar{i}_{\#J-1}] \approx I_1 \cup \dots \cup I_{\#J-1}$. We now apply IIND to the first equation in (9) and obtain

$$J = I_1 \cup \dots \cup I_{\#J} \sqsupseteq [\underline{j}, \bar{i}_{\#J-1}] \cup I_{\#J} \approx [\underline{j}, \bar{j}]$$

where the last step is by the induction hypothesis. ■

Corollary 8. \sqsupseteq satisfies IIND, IWSUB, ISM, ISD1 and ISUA, if and only if $\sqsupseteq = \sqsupseteq_{mnx}$.

Proof First note that \sqsupseteq_{mnx} satisfies these 5 axioms.

On the other hand by Lemma 1 \sqsupseteq agrees with \sqsupseteq_{mnx} on intervals. By Theorem 7 we have that for all $J \in \mathcal{I}$ that $J \approx [\underline{j}, \bar{j}]$ holds. ■

We now give one further characterization of \sqsupseteq_{mnx} . The following simple property highlights the fact that a DM comparing outcomes only focuses on a subset of all available information, see [15, p. 302-303] for a discussion on focal elements in decision making. The second axiom relates the preferences of the union two intervals with an empty intersection to focal elements.

Interval Focal Property (IFP): For all $H, J \in \mathcal{I}$ such that $H \approx J$ we have $H \cup J \approx H \cap J \approx H$.

Interval Indifference (II): For all $K, N \in INT$ such that $K \cap N = \emptyset$ we have $K \cup N \approx [\min\{\underline{k}, \underline{n}\}, \max\{\bar{k}, \bar{n}\}]$.

Theorem 9. *If \sqsupseteq satisfies IFP and II, then $J \approx [j, \bar{j}]$ for all $J \in \mathcal{I}$.*

Proof We shall proceed by induction on $\#J$. If $\#J = 1$, then this is obvious and if $\#J = 2$, then this follows from II.

If $\#J = 3$ with $J = [r, s] \cup [t, u] \cup [v, w]$ and $w \geq v > u \geq t > s \geq r$, then by the induction hypothesis

$$[r, w] \approx [r, u] \cup [v, w] \approx [r, s] \cup [t, w] .$$

Taking the intersection of $[r, u] \cup [v, w]$ with $[r, s] \cup [t, w]$ and applying IFP yields $J \approx [j, \bar{j}]$.

For $\#J \geq 4$ note that we can always obtain $J \in \mathcal{I}$ from a union of $G, H \in \mathcal{I}$ with $\#G = 3$, $\#H = \#J - 1$ and $\underline{g} = \underline{h} = \underline{j}$ and $\bar{g} = \bar{h} = \bar{j}$. By the induction hypothesis we obtain $G \approx H$. Applying IFP completes the proof.

■

Corollary 10. *\sqsupseteq satisfies II, IFP, ISM, ISD1 and ISUA, if and only if $\sqsupseteq = \sqsupseteq_{mnx}$.*

So, in the presence of ISM, ISD1 and ISUA we have that the following sets of axioms are equivalent:

- IWSUB & IMC & IROB,
- IWSUB & IIND,
- IFP & II.

The axiomatic characterizations of \sqsupseteq_{mnx} proved in this section are collected together and displayed in Table 1.

	Corollary 3	Corollary 8	Corollary 10
INTERVAL SIMPLE MONOTONICITY $r > s \Rightarrow [r] \sqsupseteq [s, r] \sqsupseteq [s]$	✓	✓	✓
INTERVAL SIMPLE DOMINANCE 1 $r > s > t \Rightarrow [t, r] \sqsupseteq [t, s]$	✓	✓	✓
INTERVAL SIMPLE UNCERTAINTY AVERSION $r > s > t \Rightarrow [s] \sqsupseteq [t, r]$	✓	✓	✓
INTERVAL WEAK SUBSTITUTION $\bar{i} < \underline{k} \Rightarrow [\bar{i}, \bar{k}] \sqsupseteq I \cup K \sqsupseteq [\underline{i}, \underline{k}]$	✓	✓	
INTERVAL MONOTONE CONSISTENCY $H \sqsupseteq J \Rightarrow H \sqsupseteq H \cup J$	✓		
INTERVAL ROBUSTNESS $H \sqsupseteq J \Rightarrow H \cup J \sqsupseteq J$	✓		
INTERVAL INDEPENDENCE $(I \sqsupseteq H \sqsupseteq J) \Rightarrow I \cup H \sqsupseteq I \cup J$		✓	
INTERVAL FOCAL PROPERTY $H \sim J \Rightarrow H \sim H \cup J \sim H \cap J$			✓
INTERVAL INDIFFERENCE $\bar{k} \geq \underline{k} > \bar{n} \geq \underline{n} \Rightarrow N \cup K \sim [\underline{n}, \bar{k}]$			✓

Table 1: Overview of axiomatic characterizations of the \sqsupseteq_{mnx} -decision rule.

5. Lexicographic Refinement

The min-max relation introduced by [15] is not fine enough to distinguish between any two acts. To refine this relation Bossert et al. introduced a *lexicographic* version, denoted by \succeq_{mnx}^L , which breaks ties by removing the worst and the best possible outcome and comparing the remainder via the min-max relation. This process is then iterated to eventually break all ties. We will here now define a lexicographic version of \sqsupseteq_{mnx}^C refining \sqsupseteq_{mnx} .⁵ The resulting refinement \sqsupseteq_{mnx}^C will not break all ties.

There are two natural ways how to translate Bossert et al.’s approach to our interval framework. Firstly, for each act $a \in \mathcal{A}$ we could remove the best and the worst possible outcome. However, this straightforward plan would not be in the spirit of the set-based approach, which we are taking here. Recall, that we argued in Section 4.1 that it should be irrelevant how many possible outcomes may yield a particular utility value u . The lexicographic refinement would then depend on whether there is only one possible outcome

⁵We follow the terminology of Bossert et al. by calling the refinement “lexicographic”. The standard use of the term lexicographic is somewhat different, see [24].

yielding the lowest utility or if there are multiple such outcomes.

The second natural way is to remove the *connected components* which contain the lowest and the highest utility. Since this approach does not suffer from the flaw highlighted above we shall pursue it here.

5.1. The Lexicographic Min-Max Relation \sqsupseteq_{mnx}^C

Definition 11. Let $J \in \mathcal{I}$ with $J = I_1 \cup I_2 \cup \dots \cup I_{\#J}$ where $I_l = [r_l, s_l]$ and $s_l < r_{l+1}$ for all l . Now put $n_J := \#J/2$, if $\#J$ is even and $n_J := (\#J + 1)/2$ if $\#J$ is odd. Let $J_0 := J$ and let for all $1 \leq t \leq n_J$ $J_t := J_{t-1} \setminus (I_t \cup I_{\#J+1-t})$ and finally we put $n_{JG} := \min\{n_J, n_G\}$ for all $J, G \in \mathcal{I}$.

For all $J, G \in \mathcal{I}$ we now define

$$J \sqsupseteq_{mnx}^C G : \iff \exists t \in \{0, \dots, n_{JG}\} \text{ such that} \tag{10}$$

$$[J_s \sim_{mnx} G_s \text{ for all } s < t] \text{ and } [J_t \sqsupseteq_{mnx} G_t \text{ or } \emptyset = G_t] .$$

So \sqsupseteq_{mnx}^C refines \sqsupseteq_{mnx} by breaking certain but not all ties. Suppose $J \approx_{mnx} G$, we then remove the *Connected Components* containing the minimum and the maximum and compare the remaining sets via \sqsupseteq_{mnx} . We do this until either a preferable set is eventually found or both J_t and G_t are empty. In this last case $J \approx_{mnx}^C G$.

If $G, J \in \mathcal{I}$, with $G \approx_{mnx} J$, are either intervals or the union of two intervals, then after removing the connected components we are left with two empty sets. Then $G \sqsupseteq_{mnx}^C J$ and $J \sqsupseteq_{mnx}^C G$, hence $G \approx_{mnx}^C J$. For such pairs $G, J \in \mathcal{I}$ the refinement \sqsupseteq_{mnx}^C does not break ties.

For example for $G = [s_1, r_1] \cup [s_2, r_2] \cup [s_3, r_3]$ with $s_1 \leq r_1 < s_2 \leq r_2 < s_3 \leq r_3$ and $J = [s_1, t_1] \cup [t_2, t_3] \cup [t_4, t_5] \cup [t_6, r_3]$ with $s_1 \leq t_1 < t_2 \leq t_3 < t_4 \leq t_5 < t_6 \leq r_3$ we have $G \approx_{mnx} J$. After removing the connected components, as described above, we are left with $[s_2, r_2]$ and $[t_2, t_3] \cup [t_4, t_5]$. Thus, $G \approx_{mnx}^C J$, if and only if $s_2 = t_2$ and $r_2 = t_5$.

For G as above and $K = [s_1, t_1] \cup [t_6, r_3]$ with $s_1 \leq t_1 < t_6 \leq r_3$ we will remove all of K and be left with the empty set. According to the definition $G \sqsupseteq_{mnx}^C K$ holds. This may be motivated as follows: At first the DM compares the worst possible outcomes and the best possible outcomes. If they are of equal preference, the DM will use the not-yet compared intervals to make a decision. Eventually, there are only intervals for one act left. Then, the DM has some idea how she appreciates these outcomes. For the other act however, there is no more interval left and the DM may thus fear the very

worst. Since we here consider a strongly risk-averse DM, a preference relation according to which some known (possible dire) consequences are strictly preferred to a state of limbo appears appropriate.

5.2. A Second Set of Axioms

We now introduce axioms to characterize \sqsupseteq_{mnx}^C . The first four are inspired by their counterparts in [15]. IUA is a variant of the previously introduced ISUA. IM is seemingly needed as we are dealing with intervals.

Interval Type 1 Dominance (ID1): For all $J \in \mathcal{I}$ and all $M, N \in INT$ such that $\underline{m} > \bar{j} \geq \underline{j} > \bar{n}$ we have $[\underline{n}, \bar{m}] \sqsupseteq N \cup J$.

Interval Extension Principle type 1 (IEP1): For all $J \in \mathcal{I}$ with $J = I_1 \cup I_2 \cup \dots \cup I_{\#J}$ and all $M, N \in INT$ with $M \cap J = N \cap J = \emptyset$ such that $I_l \sqsupseteq M \cup N$ for all $1 \leq l \leq \#J$ and such that $J \sqsupseteq M \cup N$ we have $J \cup M \cup N \sqsupseteq M \cup N$.

Interval Type 1 Monotonicity (IMON1): For all $M \in INT$ and all $J, H \in \mathcal{I}$ with $M \sqsupseteq J$ and $M \sqsupseteq H$ we have $M \sqsupseteq J \cup H$.

Interval Extension Independence (IEIND): For all $J, H \in \mathcal{I}$ and all $M, N, M', N' \in INT$ with $\bar{m} = \bar{m}'$, $\underline{n} = \underline{n}'$, $\underline{m} > \bar{j} \geq \underline{j} > \bar{n}$ and $\underline{m}' > \bar{h} \geq \underline{h} > \bar{n}'$ we have $[J \sqsupseteq H \iff J \cup M \cup N \sqsupseteq H \cup M' \cup N']$.

Interval Uncertainty Aversion (IUA): For all $N, M, K \in INT$ with $\underline{n} > \bar{m} \geq \underline{m} > \bar{k}$ we have $M \sqsupseteq K \cup N$.

Interval Monotonicity (IM): For all $r \geq s > t \geq u$ we have $[u, t] \cup [s, r] \sqsupseteq [u, t]$.

Lemma 12. *The following implications hold:*

1. $II \& ID1 \Rightarrow ISD1$,
2. $II \& ISD1 \Rightarrow IM$,
3. $ISUA \& ISM \& II \Rightarrow IUA$ and
4. $ISUA \& ISM \Rightarrow ISD2$.

Proof Proof of 1: Let $r, s, t \in \mathbb{R}$ be such that $r > s > t$. Then via ID1 and II $[t, r] \sqsupseteq [t] \cup [s] \approx [t, s]$.

Proof of 2: Let $r \geq s > t \geq u$, we have by II and ISD1 $[u, t] \cup [s, r] \approx [u, r] \sqsupseteq [u, t]$.

Proof of 3: Let $M, N, K \in INT$ with $\underline{n} < \underline{k} \leq \bar{k} < \bar{m}$. Then by ISM $K = [\underline{k}, \bar{k}] \sqsupseteq [\underline{k}]$ and via ISUA $[\underline{k}] \sqsupseteq [\underline{n}, \bar{m}]$. Furthermore, because of II we have $[\underline{n}, \bar{m}] \approx N \cup M$. Hence, $K \sqsupseteq N \cup M$.

Proof of 4: Let $r > s > t$, then by ISM we find $[s, r] \sqsupseteq [s]$ and by ISUA we have $[s] \sqsupseteq [t, r]$. This now yields $[s, r] \sqsupseteq [t, r]$. \blacksquare

5.3. A Characterization of the Lexicographic Min-Max Relation

Before giving the characterization of \sqsupseteq_{mnx}^C we first prove a lemma. The lemma sheds light on situations in which we remove connected components and are then left with the empty set.

Lemma 13. *If \sqsupseteq satisfies IUA, IEP1 and IM, then for all $J \in \mathcal{I}$ and all $M, N \in INT$ such that $\underline{m} > \bar{j}$ and such that $\underline{j} > \bar{n}$ we have $J \cup M \cup N \sqsupseteq M \cup N$.*

Proof Let $J = I_1 \cup I_2 \cup \dots \cup I_{\#J}$ with $\bar{i}_l < \underline{i}_{l+1}$ for all l . Let $M, N \in INT$ be such that $\underline{m} > \bar{j}$ and such that $\underline{j} > \bar{n}$.

Put $I_{\#J+1} := M$ and $I_0 := N$. Let $0 \leq s < l < r \leq \#J + 1$ then by IUA

$$I_l \sqsupseteq I_s \cup I_r . \quad (11)$$

Suppose that $\#J$ is even, we let $h := \#J/2$. By (11) we have $I_{h-1} \sqsupseteq I_{h+1} \cup I_{h-2}$. By application of IM we obtain $I_h \cup I_{h-1} \sqsupseteq I_{h+1} \cup I_{h-2}$. From IEP1 it follows that $I_{h-2} \cup I_{h-1} \cup I_h \cup I_{h+1} \sqsupseteq I_{h-2} \cup I_{h+1}$.

Similarly as above using IM and (11) we obtain $I_{h-2} \cup I_{h+1} \sqsupseteq I_{h-3} \cup I_{h+2}$. Using the transitivity of \sqsupseteq yields $I_{h-2} \cup I_{h-1} \cup I_h \cup I_{h+1} \sqsupseteq I_{h-3} \cup I_{h+2}$. Applying IEP1 we obtain

$$I_{h-3} \cup I_{h-2} \cup I_{h-1} \cup I_h \cup I_{h+1} \cup I_{h+2} \sqsupseteq I_{h-3} \cup I_{h+2} .$$

Continuing this way we eventually obtain $J \cup N \cup M = \bigcup_{l=0}^{\#J+1} I_l \sqsupseteq N \cup M$. Now suppose that $\#J$ is odd, we let $2g := \#J + 1$. By (11) we have $I_{g-1} \sqsupseteq I_{g-2} \cup I_g$. By IEP1 we find

$$I_{g-2} \cup I_{g-1} \cup I_g \sqsupseteq I_{g-2} \cup I_g . \quad (12)$$

Again applying (11) gives $I_{g-2} \sqsupseteq I_{g-3} \cup I_{g+1}$. By transitivity of \sqsupseteq and repeated application of IM we obtain $I_{g-2} \cup I_g \sqsupseteq I_{g-3} \cup I_{g+1}$. Together with (12) this yields $I_{g-2} \cup I_{g-1} \cup I_g \sqsupseteq I_{g-3} \cup I_{g+1}$. Hence by IEP1 and (11) we find

$$I_{g-3} \cup I_{g-2} \cup I_{g-1} \cup I_g \cup I_{g+1} \sqsupseteq I_{g-3} \cup I_{g+1} .$$

We can now follow this procedure to eventually obtain $J \cup N \cup M \sqsupseteq N \cup M$.

■

Theorem 14. \sqsupseteq satisfies ISM, ISUA, ID1, IEP1, IMON1, II and IEIND, if and only if $\sqsupseteq = \sqsupseteq_{mnx}^C$.

We shall use IM and ISD1 in this proof. Recall that IM and ISD1 follow from ID1 & II, see Lemma 12.

Proof \sqsupseteq_{mnx}^C satisfies ISM, ISUA, ID1, IEP1, II and IEIND. We shall now prove that \sqsupseteq_{mnx}^C satisfies IMON1.

Let $M \in INT$ and $J, H \in \mathcal{I}$ such that $M \sqsupseteq_{mnx}^C J$ and $M \sqsupseteq_{mnx}^C N$. Then either $(\underline{m} > \underline{j})$ or $(\underline{m} = \underline{j} \text{ and } \overline{m} > \overline{j})$. Also either $(\underline{m} > \underline{h})$ or $(\underline{m} = \underline{h} \text{ and } \overline{m} > \overline{h})$. If $\underline{m} > \underline{j}$ or if $\underline{m} > \underline{h}$ then $\underline{m} > \min\{\underline{j}, \underline{h}\}$ and hence $M \sqsupseteq_{mnx}^C J \cup H$. So we can assume that $\underline{m} = \underline{j} = \underline{h}$. Hence, $\overline{j} < \overline{m} > \overline{h}$ and thus $\overline{m} > \max\{\overline{j}, \overline{h}\}$.

Let us now turn to the other more interesting direction of the proof. It is sufficient to show for all $J, H \in \mathcal{I}$ that

$$J \approx_{mnx}^C H \text{ implies } J \approx H \quad (13)$$

and

$$J \sqsupseteq_{mnx}^C H \text{ implies } J \sqsupseteq H . \quad (14)$$

If $J \approx_{mnx}^C H$, we know that there is a t such that $J_t = H_t = \emptyset$ and $J_{t-1} \neq \emptyset \neq H_{t-1}$. Note that J_{t-1} and H_{t-1} each contain at most two connected components. Furthermore $\underline{j}_{t-1} = \underline{h}_{t-1}$ and $\overline{j}_{t-1} = \overline{h}_{t-1}$ have to hold. Hence by II $J_{t-1} \approx [\underline{j}_{t-1}, \overline{j}_{t-1}] = [\underline{h}_{t-1}, \overline{h}_{t-1}] \approx H_{t-1}$. Reading IEIND both ways yields $J_{t-2} \approx H_{t-2}$. Repeated application of IEIND eventually yields $J \approx H$.

Now suppose that $J \sqsupseteq_{mnx}^C H$. The proof proceeds by considering cases. Assuming (14) there are three possibilities:

$$J \sqsupseteq_{mnx} H \quad (15)$$

$$\exists t \in \{1, \dots, n_{JH}\} \text{ such that } J_s \approx_{mnx} H_s \text{ for all } s < t \text{ and } J_t \sqsupseteq_{mnx} H_t \quad (16)$$

$$\exists t \in \{1, \dots, n_{JH}\} \text{ such that } J_s \approx_{mnx} H_s \text{ for all } s < t \text{ and } J_t \neq \emptyset = H_t. \quad (17)$$

If (15) holds, then there are four subcases to be considered. Case A) $J, H \in INT$, case B) $J \notin INT$ and $H \in INT$, case C) $J \in INT$ and $H \notin INT$ and case D) $J, H \notin INT$.

Case A: If $\underline{j} > \underline{h}$, then by ISM and ISUA $J = [\underline{j}, \overline{j}] \sqsupseteq [\underline{j}] \sqsupseteq H$. If $\underline{j} = \underline{h}$, then $\overline{j} > \overline{h}$. By ISD1 hence $J \sqsupseteq H$.

Case B: By Lemma 13 we have $J \sqsupseteq [\underline{j}, \overline{j}]$. Also $\underline{j} > \underline{h}$ or $(\underline{j} = \underline{h} \text{ and } \overline{j} > \overline{h})$. In the first case we have $J \sqsupseteq [\underline{j}, \overline{j}] \sqsupseteq [\underline{j}]$. Since H is an interval we use ISUA

to obtain $[j] \sqsupset H$. In the second case we observe that $J \sqsupset [j, \bar{j}] \sqsupset [\underline{h}, \bar{h}]$ by the above lemma and ISD1.

Case C: Let $H := \bigcup_{l=1}^{\#H} [r_l, s_l]$ with $s_l < r_{l+1}$ for all $1 \leq l \leq \#H - 1$. There are three subcases: (X) $\underline{j} \geq \bar{h}$; (Y) $\bar{h} > \underline{j} > \underline{h}$ and (Z) $\underline{j} = \underline{h}$ and $\bar{j} > \bar{h}$.

In case CX: $J \sqsupset [j] \sqsupset [r_{\#H}, s_{\#H}]$ by ISM. Repeated application of IMON1 and II gives $J \sqsupset [r_{\#H}, s_{\#H}] \sqsupset [r_{\#H}, s_{\#H}] \cup [r_{\#H-1}, s_{\#H-1}] \sqsupset \dots \sqsupset \bigcup_{l=1}^{\#H} [r_l, s_l] = H$.

In case CY: $J \sqsupset [j]$. For all l if $\underline{j} > s_l$, then by IUA $[j] \sqsupset [r_l, s_l] \cup [r_{\#H}, s_{\#H}]$. And if $s_l \geq \underline{j} > \underline{h}$, then $[j] \sqsupset [r_1, s_1] \cup [r_l, s_l]$. Applying IMON1 gives $J \sqsupset [j] \sqsupset H$.

In case CZ: ID1 yields $[j, \bar{j}] = [\underline{h}, \bar{j}] \sqsupset H$.

Now assume (15) and case D hold. So either ($\underline{j} = \underline{h}$ and $\bar{j} > \bar{h}$) or $\underline{j} > \underline{h}$. In the the first case we obtain using Lemma 13, II and ID1: $J \sqsupset [j, \bar{j}] \approx [j] \cup [\bar{j}] \sqsupset H$. In the second case we use that $J \sqsupset [j]$ and proceed as in cases CX and CY.

In case of (16) we know that $J_t \sqsupset_{max} H_t$. We just showed that then $J_t \sqsupset H_t$ holds. Application of IEIND (possibly repeated) gives $J \sqsupset H$.

In case of (17) we know that J_{t-1} is the union of at least three pairwise disjoint intervals and that

$$J_{t-1} \neq H_{t-1} = \begin{cases} [\underline{j}_{t-1}, \bar{j}_{t-1}] \\ [\underline{j}_{t-1}, k] \cup [k + k', \bar{j}_{t-1}] \end{cases} \quad (18)$$

where $\underline{j}_{t-1} \leq k$, $k' > 0$ and $k + k' \leq \bar{j}_{t-1}$. In both cases we have $J_{t-1} \sqsupset [\underline{j}_{t-1}, \bar{j}_{t-1}] \approx H_{t-1}$ by Lemma 13 and II. Application of IEIND (possibly repeated) yields $J \sqsupset H$. ■

Although we used the axiom IM in Lemma 13 it is not required in the above characterization.

6. Utilities represented as Multi-Sets of Intervals

In our set-based approach the utility obtained from an act was taken to be the point-wise union of utilities obtained from all possible outcomes. We will now deviate from this approach and take the utility obtained from an act $a \in \mathcal{A}$ to be the union of intervals in \mathbb{R} which represent the utility obtained from the $o \in \mathcal{O}_a$. For example, if an act a has two possible outcomes which yield respective intervals $[0.5, 1.1]$ and $[0.7, 1.2]$, then the overall utility from this act is represented by $\langle [0.5, 1.1], [0.7, 1.2] \rangle$ and not by $[0.5, 1.2]$.

We thus study preference relations on such sets of intervals. Observe that if two possible outcomes o, o' of the same act a yield the same utility interval, then the multi-set of utility intervals representing the act a contains $\varphi(o)$ more than once. Using the terminology of [7] this approach could be called *vector-based*, although we do not require the vectors we compare to be of the same dimension nor does the order of the components of the vectors matter. We hence call our approach *multi-set based*.

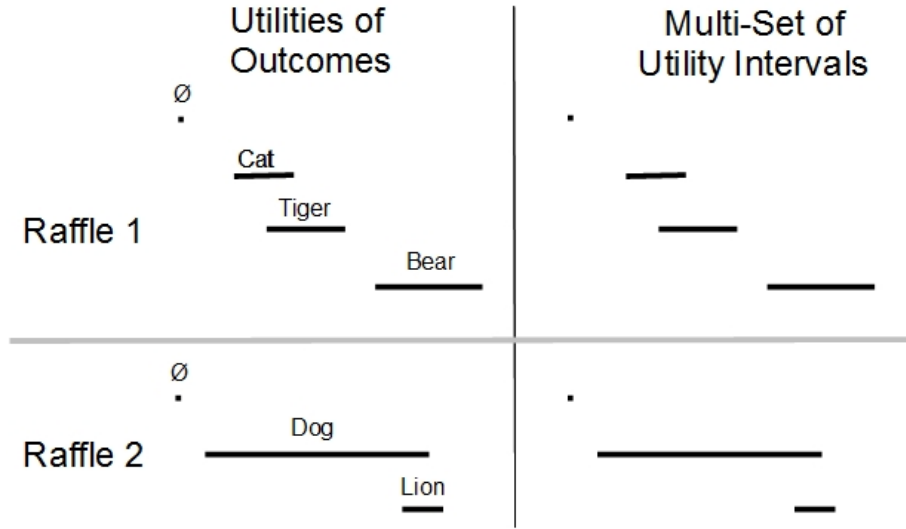


Figure 2: Multi-set aggregation of utility intervals arising from a decision problem between two raffles.

With the sole exception of IDOUB all axioms we introduce in this section are direct translations of axioms in [15] into our framework and thus inherit the justification of their brethren. Again, we shall assume that the DM has a linear preference relation over single outcomes. This now amounts to a binary relation over INT , which we denote by \sqsupseteq .

Definition 15. Let $V = \{I_1, I_2, \dots, I_n\}$ with $I_i \in INT$ be a finite multi-set of intervals; the I_i are thus not all necessarily pairwise distinct. We let \mathcal{SI} be the set of all such multi-sets of intervals. For $V \in \mathcal{SI}$ let $\#V$ denote the number of intervals comprising V .

Definition 16. For $V \in \mathcal{SI}$ let \underline{v} be the interval I in V such that for all other $J \in V$ we have $J \sqsupseteq_{max} I$. Let \bar{v} be the so defined maximum. Define

the strict part of the min-max decision rule \sqsupseteq_{2mnx} on \mathcal{SI} for $V, W \in \mathcal{SI}$ by

$$V \sqsupseteq_{2mnx} W \text{ if and only if } [\underline{v} \sqsupseteq_{mnx} \underline{w} \text{ or } (\underline{v} = \underline{w} \text{ and } \bar{v} \sqsupseteq_{mnx} \bar{w})] . \quad (19)$$

Recall that for $I, J \in INT$ we have $I = J$, if and only if $I \approx_{mnx} J$. We can thus reorder $V = \{I_1, I_2, \dots, I_{\#V}\}$ to achieve that for all l we have $I_{l+1} \sqsupseteq_{2mnx} I_l$ with $I_l \approx_{2mnx} I_{l+1}$, if and only if $I_l = I_{l+1}$. For $V = \{I_1, I_2, \dots, I_{\#V}\} \in \mathcal{SI}$ and $W = \{J_1, J_2, \dots, J_{\#W}\} \in \mathcal{SI}$ the relation \sqsupseteq_{2mnx} is given by first comparing the worst interval in V to the worst interval in W via the min-max relation. If $I_1 \sqsupseteq_{mnx} J_1$, then $V \sqsupseteq_{2mnx} W$. If $I_1 = J_1$, then $I_{\#V}$ and $J_{\#W}$ are compared via the min-max rule. In that case if $I_{\#V} \sqsupseteq_{mnx} J_{\#W}$, then $V \sqsupseteq_{2mnx} W$. Thus, $V \approx_{mnx} J$ holds, if and only if $I_1 = J_1$ and $I_{\#V} = J_{\#W}$.

In plain English, \sqsupseteq_{2mnx} compares worst possible outcomes by the min-max rule. If there is a tie, best possible outcomes are compared by the min-max rule. If there is a second tie, then $V \approx_{2mnx} W$. The double application of the min-max rule inspired the name \sqsupseteq_{2mnx} . Likewise, the number “2” in the following axioms indicates that they are closely related to their cousins in the set-based interval approach.

For the example in Figure 2, a DM applying \sqsupseteq_{2mnx} would prefer a ticket of raffle 2 over a ticket of raffle 1. Both tickets have the same worst possible outcome, however for the best possible outcomes according to the min-max rule (winning a “bear” or a “lion”) the outcome “lion” is preferred to the outcome “bear” according to the min-max rule. Recall that a DM applying \sqsupseteq_{mnx} prefers a ticket of raffle 1.

Interval Cautious Substitution (ICSUB): Let $V \in \mathcal{SI}$, $I, J \in INT$ be such that $I, J \notin V$ and $J \sqsupseteq I$, then $J \cup V \sqsupseteq V \cup I$.⁶

Interval Double (IDOUB): Let $V \in \mathcal{SI}$, $I, J \in INT$ and $I, J \in V$ be such that $I = J$, then $V \approx V \setminus I$.

2 Interval Simple Monotonicity (2ISM): For all $I, J \in INT$ such that $I \sqsupseteq J$ we have $I \sqsupseteq I \cup J \sqsupseteq J$.

2 Interval Simple Dominance 1 (2ISD1): For all $I, J, K \in INT$ such that $I \sqsupseteq J \sqsupseteq K$ we have $K \cup I \sqsupseteq K \cup J$.

2 Interval Simple Uncertainty Aversion (2ISUA): For all $I, J, K \in INT$ such that $I \sqsupseteq J \sqsupseteq K$ we have $J \sqsupseteq I \cup K$.

⁶Here and in the following we drop the brackets $\{\cdot\}$ in the notation. So $V \cup I$ is the multi-set containing the interval I and the intervals in V .

The axiom IDOUB implies that multiple outcomes from the same act which yield the exact same utility interval are treated as if there was only one such outcome. Accepting this axioms is appropriate in case the DM is only interested in which utility intervals $I \in INT$ an act $a \in \mathcal{A}$ may possibly yield.

Theorem 17. \sqsupseteq satisfies ICSUB, IDOUB, ISM, 2ISM, ISD1, 2ISD1, ISUA and 2ISUA, if and only if $\sqsupseteq = \sqsupseteq_{2mnx}$.

Proof Clearly, \sqsupseteq_{2mnx} satisfies these axioms.

Now suppose that \sqsupseteq on \mathcal{SI} satisfies these axioms. By the proof of Lemma 1 \sqsupseteq agrees with \sqsupseteq_{mnx} on INT . Since \sqsupseteq_{mnx} agrees with \sqsupseteq_{2mnx} on INT we have that \sqsupseteq agrees with \sqsupseteq_{2mnx} on INT .

Next consider a $V \in \mathcal{SI}$. By replacing all intervals in V different from \underline{v} with \bar{v} and applying ICSUB and IDOUB we obtain $\underline{v} \cup \bar{v} \sqsupseteq V$. Replacing in V all intervals other than the maximum with \underline{v} and applying ICSUB and IDOUB we obtain $V \sqsupseteq \underline{v} \cup \bar{v}$. Thus $V \approx \underline{v} \cup \bar{v}$. For the remainder of the proof we may thus suppose that $V = \underline{v} \cup \bar{v}$ and $W = \underline{w} \cup \bar{w}$, possibly $\underline{v} = \bar{v}$ and/or $\underline{w} = \bar{w}$. We will now imitate the proof of Lemma 1.

If i) $V \approx_{2mnx} W$ then $\underline{v} \approx_{mnx} \underline{w}$ and $\bar{v} \approx_{mnx} \bar{w}$. Thus $\underline{v} = \underline{w}$ and $\bar{v} = \bar{w}$ and hence $V = W$. $V \approx W$ follows trivially.

If ii) $V \sqsupset_{2mnx} W$, then either a) $\underline{v} = \bar{v}$ and $\underline{w} = \bar{w}$, b) $\underline{v} \neq \bar{v}$ and $\underline{w} = \bar{w}$, c) $\underline{v} = \bar{v}$ and $\underline{w} \neq \bar{w}$, d) $\underline{v} \neq \bar{v}$ and $\underline{w} \neq \bar{w}$.

Suppose a) holds, then $V = \underline{v} \sqsupset_{mnx} \underline{w} = W$, hence $V \sqsupset W$.

Suppose b) holds, if $\underline{v} = W$ then by 2ISM $V = \underline{v} \cup \bar{v} \sqsupset W$. If $\underline{v} \neq W$, then $\underline{v} \sqsupset W$; since we assumed that $V \sqsupset_{2mnx} W$. We thus have $V \sqsupseteq \underline{v} \sqsupset W$.

Suppose c) holds, then $V = \underline{v} \sqsupset \underline{w}$. If $\underline{v} \sqsupseteq \bar{w} \sqsupset \underline{w}$, then by 2ISM $V = \underline{v} \sqsupseteq \bar{w} \sqsupset \underline{w} \cup \bar{w} = W$. If $\bar{w} \sqsupseteq \underline{v} \sqsupset \underline{w}$, then by 2ISUA we have $V \sqsupset W$.

Suppose d) holds and recall that we assume that $V \sqsupset_{2mnx} W$. One of the following then has to hold

$$\bar{v} \sqsupset \underline{v} \sqsupset \bar{w} \sqsupset \underline{w}, \quad (20)$$

$$\bar{v} \sqsupset \underline{v} = \bar{w} \sqsupset \underline{w}, \quad (21)$$

$$\bar{v} \sqsupset \underline{v} \text{ and } \bar{w} \sqsupset \underline{v} \sqsupset \underline{w}, \quad (22)$$

$$\bar{v} \sqsupset \bar{w} \sqsupset \underline{v} = \underline{w}. \quad (23)$$

If (20) holds, then by 2ISM $V = \bar{v} \cup \underline{v} \sqsupset \underline{v} \sqsupset \bar{w} \sqsupset \bar{w} \cup \underline{w} = W$.

If (21) holds, then by 2ISM $V = \bar{v} \cup \underline{v} \sqsupset \underline{v} = \bar{w} \sqsupset \bar{w} \cup \underline{w} = W$.

If (22) holds, then by 2ISM and 2ISUA $V = \underline{v} \cup \bar{v} \sqsupset \underline{v} \sqsupset \underline{w} \cup \bar{w} = W$.

If (23) holds, then simply by 2ISD1 $V \sqsupset W$. ■

6.1. Lexicographic Refinement

Defining a Bossert-style lexicographic refinement \sqsupset_{2mnx}^L of \sqsupset_{2mnx} on \mathcal{SI} is uncontentious, since the worst and best element of a $V \in \mathcal{SI}$ are well-defined. Instead of giving a technical definition of \sqsupset_{2mnx}^L , which would require the introduction of more notation, we give an intuitive description of \sqsupset_{2mnx}^L .

\sqsupset_{2mnx}^L is defined as a refinement of \sqsupset_{2mnx} . Suppose $V, W \in \mathcal{SI}$ are such that $V \approx_{2mnx} W$ and $V \neq W$. Then we remove the worst and the best element of V and of W , thus we obtain V' and W' ; where worst and best are here according to \sqsupset_{mnx} . We then compare the V' to W' via \sqsupset_{2mnx} . If W' is empty, then $V \sqsupset_{2mnx}^L W$. If $V', W' \neq \emptyset$, then V' is compared to W' via \sqsupset_{2mnx} . If $V' \sqsupset_{2mnx} W'$, then $V \sqsupset_{2mnx}^L W$. If $V' \approx_{2mnx} W'$, then we again remove the worst and best elements and continue in this manner.

Since we assumed that $V \neq W$ we eventually obtain either $V \sqsupset_{2mnx}^L W$ or $W \sqsupset_{2mnx}^L V$, that is according to \sqsupset_{2mnx}^L no two different $V, W \in \mathcal{SI}$ are equally preferred. We now give an axiomatic characterization of \sqsupset_{2mnx}^L .

Definition 18. Let $\mathcal{SI}_2 := \{V \in \mathcal{SI} | \#V \leq 2\}$.

Lemma 19. If \sqsupset satisfies ISM, 2ISM, ISD1, 2ISD1, ISUA and 2ISUA, then \sqsupset agrees with \sqsupset_{2mnx}^L on \mathcal{SI}_2 .

Proof Lemma 1 shows that \sqsupset agrees with \sqsupset_{mnx} on INT and hence it agrees there with \sqsupset_{2mnx}^L . We now mimic the proof of Lemma 2 on page 303 in [15] to extend the agreement to \mathcal{SI}_2 . ■

Completely analogously to Lemma 4 on page 308 in [15] we will obtain the next lemma. Before that we need to introduce one further axiom.

2 Interval Extension Principle 1 (2IEP1): For all $V \in \mathcal{SI}$, $N, M \in INT$ such that $N, M \notin V$ and such that for all $v \in V$ we have $N \sqsupset v \sqsupset M$ and $V \sqsupset N \cup M$, then $V \cup N \cup M \sqsupset N \cup M$.

Lemma 20. Let $V \in \mathcal{SI}$, $M, N \in INT$ be such that $M, N \notin V$ and let \sqsupset satisfy 2ISM, 2ISUA and 2IEP1. If for all $I \in V$ we have $M \sqsupset I \sqsupset N$, then $V \cup M \cup N \sqsupset M \cup N$.

To complete the axiomatic characterization we need to introduce a few more axioms.

2 Interval Dominance 1 (2ID1): Let $V \in \mathcal{SI}$ and $N, M \in INT$ be such that for all $I \in V$ we have $N \sqsupset I \sqsupset M$. Then $N \cup M \sqsupset V \cup M$.

2 Interval Monotonicity 1 (2IMON1): Let $N \in INT$, $V, W \in \mathcal{SI}$ be such that $N \sqsupset V$ and $N \sqsupset W$. Then $N \sqsupset V \cup W$.

2 Interval Extension Independence (2IEIND): Let $N, M \in INT$, $V, W \in \mathcal{SI}$ be such that $M, N \notin V, W$ and such that $N \sqsupset I$ and $J \sqsupset M$ for all $I \in V$ and all $J \in W$. Then, $V \sqsupset W$ holds, if and only if $N \cup V \cup M \sqsupset N \cup W \cup M$ holds.

Theorem 21. *If \sqsupseteq satisfies ISM, 2ISM, ISUA, 2ISUA, ISD1, 2ID1, 2IEP1, 2IMON1 and 2IEIND, then $\sqsupseteq = \sqsupseteq_{2mnx}^L$.*

Proof First note that \sqsupseteq_{2mnx}^L satisfies all these axioms.

For the other direction note that we already proved that \sqsupseteq agrees with \sqsupseteq_{2mnx}^L on \mathcal{SI}_2 . The rest of the proof is a simple adaptation of the proof of Theorem 5 on page 309 in [15]. \blacksquare

Comparing the characterization of \sqsupseteq_{2mnx}^L to that of \succeq_{mnx}^L (see Theorem 5 in [15]) we note that SM, SUA and D1 have all been split into two axioms reflecting the fact that we first have to fix the DM's preferences on INT before we can move to comparisons of multi-sets of intervals. EP1, MON1 and EIND on the other hand have been translated directly to the here introduced framework.

7. Conclusions

We have put forward two frameworks for choice under complete uncertainty. The key ingredient in these frameworks is the assumption that the ordinal utility obtained from an outcome cannot be described by a simple number but is better represented by an interval of utilities. The first framework applied a set-based approach to aggregate utilities from different possible outcomes from the same act while the second framework relied on multi-sets to aggregate uncertain utilities. Then we axiomatically characterized several decision rules for risk-averse boundedly rational DMs in these frameworks.

Our approach is limited by the assumptions we have made. For instance we assumed that there is absolutely no information on the (relative) likelihood of possible outcomes available nor can utilities be cardinally compared.

The further assumption of transitivity of preference relations for decision problems under (complete) uncertainty seems to be in line with human DMs facing such problems. Vrijdags found in [45] that human DMs faced with a choice under complete uncertainty rarely violate transitivity. A related similar result is that of Birnbaum & Schmidt [10] who reported that human DMs display transitive preferences when facing risky choices. The transitivity assumption in our approach hence appears to be in line with real-world human DMs.

Transferring other preference relations (for instance median based relations) to our framework and (axiomatic) investigations of these transferred relations are logical next steps to take. This could yield a better understanding of the frameworks introduced here as well as [the properties of] their preference relations. Furthermore, empirical investigations of choice under complete or imprecise uncertainty (such as [31, 45]) may be carried out to determine whether decision rules based on our frameworks can lead to better explanations of observed choice behaviour under complete uncertainty (or even to predict such choices).

Appendix A. A Condensed Extract of Bossert, Pattanaik & Xu and Arlegi

Appendix A.1. The Framework

Let X denote the finite set of possible outcomes, where $\#X$ denotes the size of X . Let \mathcal{K} be the power set of X without the empty set. A subset of X containing at least two elements is interpreted as the uncertain prospect with possible outcomes “being” the elements of this subset. A singleton set $\{x\} \subset X$ is interpreted as a “trivial” uncertain (i.e. certain) prospect with only one possible outcome, namely x . Put also $\mathcal{K}_2 := \{A \in \mathcal{K} \mid 1 \leq \#A \leq 2\}$.

Furthermore, consider a fixed linear preference ordering R on X , i.e. R is reflexive, transitive and complete. For $A \in \mathcal{K}$ let \underline{a}, \bar{a} denote, respectively the minimum and maximum of A with respect to R . Let P be the antisymmetric part of R and I be the symmetric part.

Let \succeq be an ordering over \mathcal{K} , i.e. it is reflexive, transitive and complete. This ordering is interpreted as the agent’s preference over the uncertain outcomes. \succ and \sim denote, respectively, the asymmetric and symmetric parts of \succeq .

Appendix A.2. Existing Axiomatic Characterization of classical Min-Max Relation

The following axioms were introduced in [4] and [15].

Simple Monotonicity (SM): For all $x, y \in X$ such that xPy we have $\{x\} \succ \{x, y\} \succ \{y\}$.

Simple Dominance 1 (SD1): For all $x, y, z \in X$ such that $xPyPz$ ⁷ we have $\{x, z\} \succ \{y, z\}$.

Simple Dominance 2 (SD2): For all $x, y, z \in X$ such that $xPyPz$ we have $\{x, y\} \succ \{x, z\}$.

Simple Uncertainty Aversion (SUA): For all $x, y, z \in X$ such that $xPyPz$ we have $\{y\} \succ \{x, z\}$.

Simple Uncertainty Appeal (SUP): For all $x, y, z \in X$ such that $xPyPz$ we have $\{x, z\} \succ \{y\}$.

Substitution (SUB): For all $A \in \mathcal{K}$, for all $y \in A$ and all $x \in X \setminus A$ with xPy we have $(A \cup \{x\}) \setminus \{y\} \succeq A$.

Monotone Consistency (MC): For all $A, B \in \mathcal{K}$ with $A \succeq B$ we have $A \cup B \succeq B$.

Robustness (ROB): For all $A, B, C \in \mathcal{K}$ with $A \succeq B$ and $A \succeq C$ we have $A \succeq B \cup C$.

The two basic preference relations investigated are the min-max relation \succeq_{mnx} and the max-min relation \succeq_{mxn} . They are defined on \mathcal{K} as follows

$$A \succeq_{mnx} B, \text{ if and only if } [\underline{a}P\underline{b} \text{ or } (\underline{a}I\underline{b} \text{ and } \bar{a}R\bar{b})] \quad (\text{A.1})$$

$$A \succeq_{mxn} B, \text{ if and only if } [\bar{a}P\bar{b} \text{ or } (\bar{a}I\bar{b} \text{ and } \underline{a}R\underline{b})] . \quad (\text{A.2})$$

In Lemma 2 on pages 303-304 in [15] the authors prove two equivalences \succeq satisfies SM, SD1 and SUA, if and only if \succeq agrees with \succeq_{mnx} on \mathcal{K}_2 .

\succeq satisfies SM, SD2 and SUP, if and only if \succeq agrees with \succeq_{mxn} on \mathcal{K}_2 .

Theorem 1 on page 222 in [4] reads:

If \succeq satisfies SUB, MC and ROB, then $A \sim \{\underline{a}, \bar{a}\}$ for all $A \in \mathcal{K}$.

Theorem 2 in [4] on page 223 states that

⁷This standard abuse notation, formally correct and cumbersome notation is (xPy and yPz). We will continue to abuse the notation similarly elsewhere.

\succeq satisfies SM, ROB, MC, SD1 and SUA, if and only if $\succeq = \succeq_{mnx}$.
 \succeq satisfies SM, ROB, MC, SD2 and SUP, if and only if $\succeq = \succeq_{m xn}$.

Appendix B. Improved and further Characterizations

It was later found that the axiomatic characterization of the min-max relation given in [15] was erroneous and subsequently a correct characterization was given, see [4]. For this later characterization three new axioms were introduced: SUB, MC and ROB.

Appendix B.1. Improved Characterization of the classical Min-Max Relation

We will here show how to characterize \succeq_{mnx} by only slightly modifying one of the axioms in the erroneous proof.

Independence' (IND'): For all $A, B \in \mathcal{K}$ and all $x \in X$ such that $\{x\} \succ A \succeq B$ we have $A \cup \{x\} \succeq B \cup \{x\}$.

Theorem 22. *If \succeq satisfies SM, IND' and SD1, then for all $A \in \mathcal{K}$ we have $A \sim \{\underline{a}, \bar{a}\}$.*

Proof The proof is done by induction. If $\#A \leq 2$, then clearly $A \sim \{\underline{a}, \bar{a}\}$. Let $A = \{b_1, \dots, b_m\}$ with $b_1 P b_2 P \dots P b_{m-1} P b_m$. If $\#A = 3$, then by SM

$$\{b_2, b_3\} \succ \{b_3\} . \quad (\text{B.1})$$

Since by SM we have

$$\{b_1\} \succ \{b_1, b_2\} \succ \{b_2\} \succ \{b_2, b_3\} \succ \{b_3\} \quad (\text{B.2})$$

we can apply IND' to find

$$\{b_1, b_2, b_3\} \succeq \{b_1, b_3\} . \quad (\text{B.3})$$

Now for the other direction we have by SM and SD1

$$\{b_1\} \succ \{b_1, b_3\} \succ \{b_2, b_3\} . \quad (\text{B.4})$$

and via IND' we find

$$\{b_1, b_3\} \succeq \{b_1, b_2, b_3\} . \quad (\text{B.5})$$

Hence, for $\#A \leq 3$ we have $A \sim \{\underline{a}, \bar{a}\}$.

Now assume that $m := \#A \geq 4$. By the induction hypothesis, SM and SD1 we find

$$\{b_1\} \succ \{b_2\} \succ \{b_2, b_m\} \sim \{b_2, \dots, b_m\} . \quad (\text{B.6})$$

Applying IND' twice gives

$$\{b_1, b_2, b_m\} \sim \{b_1, b_2, \dots, b_m\} . \quad (\text{B.7})$$

Using the induction hypothesis on the left hand side yields

$$\{b_1, b_m\} \sim \{b_1, b_2, b_m\} \sim \{b_1, b_2, \dots, b_m\} . \quad (\text{B.8})$$

■

We now restate Lemma 2 in [15] on page 303.

Lemma 23. \succeq satisfies SM, SD1 and SUA, if and only if \succeq agrees with \succeq_{mnx} on \mathcal{K}_2 .

Theorem 24. \succ satisfies SM, SD1, SUA and IND', if and only if $\succeq = \succeq_{mnx}$.

Proof First of all let us verify that \succeq_{mnx} satisfies IND'.

Let $\{x\} \succ_{mnx} A$ and $A \succeq_{mnx} B$. If $\#A = 1$, then $x P \underline{a}$. If furthermore $A \sim_{mnx} B$ then $A = B$ and hence $A \cup \{x\} = B \cup \{x\}$ and so $A \cup \{x\} \sim_{mnx} B \cup \{x\}$. If $A \succ_{mnx} B$, then $\underline{a} P \underline{b}$ and so $\{\underline{a}\} \cup \{x\} P \{\underline{b}\} \cup \{x\}$. Hence $A \cup \{x\} \succ_{mnx} B \cup \{x\}$.

If $\#A \geq 2$, then $x P \underline{a}$. Since $\underline{a} R \underline{b}$ we have $x P \underline{b}$. This yields $A \cup \{x\} R B \cup \{x\}$. If $\underline{a} = \underline{b}$, then $\bar{a} R \bar{b}$. Then also $A \cup \{x\} R B \cup \{x\}$. We hence have $A \cup \{x\} \succeq_{mnx} B \cup \{x\}$.

Hence, \succ_{mnx} satisfies IND' and it also satisfies the other above axioms.

Combining Theorem 22 and Lemma 23 yields the other direction. ■

Appendix B.2. Further Characterizations

We now translate the ideas from Section 4.6 to this framework.

Focal Property (FP): For all $A, B \in \mathcal{K}$ with $A \sim B$ we have $A \sim A \cup B \sim A \cap B$.

Forgotten Middle (FM): For all $A \in \mathcal{K}$ with $A = \{b_1, b_2, b_3\}$ with $b_1 P b_2 P b_3$ we have $A \sim \{b_1, b_3\}$.

Lemma 25. If \succeq satisfies FP and FM, then for all $A \in \mathcal{K}$ we have $A \sim \{\underline{a}, \bar{a}\}$.

Proof The proof is by induction on $\#A$. If $\#A \leq 2$, then there is nothing to prove.

If $\#A = 3$, we apply FM.

If $\#A =: m \geq 4$ with $A = \{b_1, b_2, \dots, b_m\}$ and $b_1 P b_2 P b_3 \dots b_{m-1} P b_m$, then $A = \{b_1, b_2, b_m\} \cup \{b_1, b_3, b_4, \dots, b_m\}$. Note that by the induction hypothesis these last two sets are of equal preference. Applying FP to their union and intersection yields $A \sim \{a, \bar{a}\}$. ■

Theorem 26. \succeq satisfies SM, SD1, SUA, FP and FM, if and only if $\succeq = \succeq_{mnx}$.

In Table B.2 the characterizations given in the appendix can be seen at one glance. Unsurprisingly, there is a high degree of symmetry between Table 1 and Table B.2.

	"Theorem 3" in [15]	Theorem 2 in [4]	Theorem 24	Theorem 26
SIMPLE MONOTONICITY $xPy \Rightarrow \{x\} \succ \{x, y\} \succ \{y\}$	X	✓	✓	✓
SIMPLE DOMINANCE 1 $xPyPz \Rightarrow \{x, z\} \succ \{y, z\}$	X	✓	✓	✓
SIMPLE UNCERTAINTY AVERSION $xPyPz \Rightarrow \{y\} \succ \{x, z\}$	X	✓	✓	✓
INDEPENDENCE $(x \notin A \cup B \ \& \ A \succ B) \Rightarrow A \cup \{x\} \succeq B \cup \{x\}$	X			
INDEPENDENCE' $\{x\} \succ A \succeq B \Rightarrow A \cup \{x\} \succeq B \cup \{x\}$			✓	
MONOTONE CONSISTENCY $(A \succeq B \ \& \ A \succeq C) \Rightarrow A \succeq B \cup C$		✓		
ROBUSTNESS $A \succeq B \Rightarrow A \cup B \succeq B$		✓		
FOCAL PROPERTY $A \sim B \Rightarrow A \sim A \cup B \sim A \cap B$				✓
FORGOTTEN MIDDLE $xPyPz \Rightarrow \{x, z\} \sim \{x, y, z\}$				✓

Table B.2: Schematic overview of axiomatic characterizations of the \succeq_{mnx} -decision rule. The quotation marks around Theorem 3 indicate an erroneous proof.

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Acknowledgments

Section 6 relies on an idea suggested to the author by Guillaume Aucher. I am also heavily indebted to Laurent Lardon, Jean-Philippe Steyer, Ricardo Büttner and Jon Williamson. Many thanks to Teddy Groves for proof-reading and last but not least to three anonymous referees; all mistakes that remain are entirely mine.

The idea for this study and the main part of the work was carried out while the author was at the LBE in Narbonne (France) supported by a grant from the Environment & Agronomy Department of INRA. Work continued on this study while the author worked at the FOM in Munich (Germany) where he was supported by grant 17103X10 from the German Federal Ministry of Education and Research. The work was completed while the author was (and currently is) supported by the AHRC funded project *From objective Bayesian epistemology to inductive logic* at the University of Kent (United Kingdom).