Conditional beliefs aren't conditional probabilities

The claim that conditional rational degrees of belief are conditional probabilities is falsified by the following simple counterexample.

Red Faces. Suppose that a fair six-sided die is to be rolled (proposition X) and that each face of the die is coloured red, blue or green (E). Consider the outcome that the number rolled will be three or greater (A). It is reasonable to believe A to degree $\frac{2}{3}$: given that the die is fair, each number has chance $\frac{1}{6}$ of being rolled, and four of the 6 numbers on the die are greater than or equal to three. So, if conditional degrees of belief are conditional probabilities, it is rationally permissible to set:

$$P(A|XE) = 2/3.$$

Now consider an alternative outcome: that the colour rolled (i.e., the colour of the uppermost face) is red (R). It is clearly reasonable to believe R to degree $\frac{1}{3}$, on the grounds that red is one out of the three possible colours and there is no evidence that favours one of these colours over any of the others. Thus it is permissible to set:

$$P(R|XE) = 1/3.$$

Now suppose in addition that the red faces are precisely those that are numbered three or greater, i.e., $A \leftrightarrow R$. Given that the die is fair, it is again clearly rationally permissible to believe A to degree $\frac{2}{3}$:

$$P(A|XE(A \leftrightarrow R)) = 2/3.$$

(Note that for these conditional probabilities to be well defined, it must be rationally permissible to set $P(XE(A \leftrightarrow R)) > 0$, i.e., to assign some positive credence to the claim that the die is fair, faces 3-6 are red and faces 1-2 are blue or green.)

It turns out, however, that these assignments of degree of belief are inconsistent: there is no probability function that satisfies them all (Wallmann & Williamson 2020: *EJPS* 10(3); Williamson 2023: *IJB* 19(2), 295–307). It is thus not possible to use conditional probabilities to validate the above judgements about rational permissibility: i.e., conditional beliefs are not always identifiable with conditional probabilities.

Consequences. Let belief function B represent a rationally permissible assignment of conditional degrees of belief: $B_C(A)$ is the degree to which proposition A is believed under condition C, for all A and C in a given domain of propositions. The claim that conditional beliefs are conditional probabilities can be formulated as follows:

CBCP. For any belief function B, there is some probability function P such that $B_C(A) = P(A|C)$ for all A and C.

In the red faces counterexample we have an assignment of degrees of belief that is clearly rationally permissible, yet cannot be captured by a conditional probability function. Hence CBCP is false.

This has two important consequences.

Firstly, if CBCP is taken to be constitutive of Bayesianism, as is standardly the case, then Bayesianism is untenable. The red faces problem threatens the tools of Bayesianism as well as its philosophical foundations. Bayes' Theorem is only of use if conditional probabilities are themselves of use, but this requires some connection between conditional probabilities and rational belief such as CBCP. Bayesian conditionalisation also apparently rests on CBCP: why update by means of conditional probabilities unless those conditional probabilities represent degrees of belief conditional on new evidence? Without Bayes' Theorem or Bayesian conditionalisation, Bayesianism would seem very impoverished.

Second, the 'new paradigm' in the psychology of reasoning, which seeks to understand our reasoning by appeal to conditional probabilities, is untenable without CBCP or something like it (Oaksford & Chater 2020: *ARP* 71(1), 305–330). For instance, the new paradigm analyses our use of conditional propositions in terms of conditional probabilities. This analysis involves two steps: an appeal to conditional beliefs to analyse cognition involving conditional propositions and then an application of CBCP to connect to conditional probability. Without CBCP, this analysis cannot succeed.

A Potential Resolution. The red faces problem shows that conditional beliefs can't always be construed as conditional probabilities. On the other hand, the successes of Bayesianism and of the new paradigm show that it can sometimes be helpful to identify conditional beliefs with conditional probabilities. What we need is a more fundamental theory to explain the successes and failures of CBCP.

There is a non-standard approach to Bayesianism that might help here (Williamson 2010: *In defence of objective Bayesianism*, OUP). This version of Bayesianism identifies conditional beliefs with probabilities, but not conditional probabilities:

CBP. For any belief function B and proposition C, there is some probability function P_C such that $B_C(A) = P_C(A)$ for all A.

How are these unconditional probabilities obtained? Firstly, P_C must satisfy constraints imposed by C—in particular, constraints imposed by calibration to chances: if one establishes from C that the chance of A is x then $P_C(A) = x$, as long as C doesn't imply anything that defeats this ascription (e.g., proposition A itself). Second, P_C should be maximally equivocal with respect to propositions whose probability isn't determined by constraints imposed by C. This is typically explicated by setting P_C to be the function, from all those that satisfy constraints imposed by C, that has maximal entropy.

This version of Bayesianism is immune to the red faces problem: it will consistently set $P_{XE}(A) = 2/3$ (by calibrating to the chance information in X), $P_{XE}(R) = 1/3$ (equivocating between the three possible colours), and $P_{XE(A \leftrightarrow R)}(A) = 2/3$ (by calibration to chance again).

The theory can also help to explain when it is safe to conditionalise. If (i) learning D only imposes the constraint $P(D) = 1$, (ii) $P_C(D) > 0$, and (iii) $P_C(\cdot|D)$ satisfies all the constraints imposed by C, then it is safe to conditionalise on D, i.e., $P_{CD}(\cdot) = P_C(\cdot|D)$; see Result 1 of Seidenfeld (1986: Entropy and Uncertainty, *PoS* 53: 467–491) and Theorem 5.16 of Williamson (2017: *Lectures on inductive logic*, OUP). In the red faces problem, it is not safe to conditionalise on $A \leftrightarrow R$

because $P_{XE}(\cdot|A \leftrightarrow R)$ does not satisfy all the constraints imposed by XE. In particular, as the Appendix of Williamson (2023) shows, $P_{XE}(A|A \leftrightarrow R) = 1/2 \neq 2/3$, the value required by calibration to the chance information in XE.

Thus, although this version of Bayesianism may seem unorthodox, it is explanatory. In any case, a significant departure from Bayesian orthodoxy is required to avoid *red faces*.

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The Reasoner Speculates

Benefits of cybernetic models in philosophy

A common research method among philosophers is the usage of thought experiments. Take for example John Searle's 'Chinese Room' or Frank Jackson's 'Mary's Room' argument. David Lewis goes further by using neuron diagrams to represent causality in his counterfactual theories of causation. His method has since been further refined and developed. Interestingly, the usage of logical circuits or finite automaton to represent causal relations has not yet been considered. As an advantage, the latter can be visualised in cyberspace using spreadsheets and tested in practice. Furthermore, it is not only in the problem of causality that cybernetic models can fruitfully be used to provide a philosophical explanation, they can also be utilised to represent logical semantic problems. Let us consider an example for this.

Many logic handbooks allude to the obvious connection between propositional logic and logic circuits. Truth functions in logic can be represented by logic circuits in which the high or low voltage levels of the circuits correspond to the true and false logic values, respectively. At the propositional logic level, the logical connectives of propositions can be simulated by logic circuits as follows: the true or false logical evaluation of atomic propositions corresponds to the high or low level of the circuit input and the truth value of compound propositions corresponds to the circuit output state. A high circuit output signifies that the compound sentence is evaluated as true, whereas a low output indicates that the compound sentence is false. It is well known that in the world of logic circuits, the AND connective in logic corresponds to the AND gate, the OR connective to the OR gate and the negation operation to the inverter. The output of a circuit equivalent to contradiction is always low and that of a circuit corresponding to tautology is always high irrespective of the input state. The remaining compound formulas correspond to logic circuits with a high output level for some inputs and low output level for other inputs. However, what logical circuit can model a circular sentence?

Indeed, every formula in propositional calculus can be modelled based on an equivalent logic circuit, specifically referred to as a combinational logic circuit. However, not all logic circuits are combinational logic circuits. The range of logic circuits is wider than that of the combinational logic circuits. It includes logic circuits whose input states do not determine unambiguously their output states, i.e. the output is not a function of the input. This is because the circuit has feedback. Circuits that contain feedback are called sequential logic circuits. Although every formula in propositional calculus can be modelled based on an equivalent combinational logic circuit, it remains unclear whether the converse theorem is valid. Can every sequential logic circuit be equivalent to a formula in propositional calculus? Does any formula at the propositional logic level correspond to sequential logic circuits?

Sequential logic circuits have memory owing to feedback mechanisms. (The operation of these circuits is mathematically isomorphic to that of a finite automaton. Examples of such circuits include flip-flops, registers, counters, clocks and memories.) The output state of sequential logic circuits is not a function of the input states but depends on previous input states. In contrast, the truth value of formulas in propositional calculus is a function of the evaluation of atomic formulas, without considering previous evaluations of these formulas. Therefore, the answer is negative; logical formulas cannot be simply matched with sequential logic circuits at the propositional logic level. However, logical relations between sentences may exist beyond propositional logic, corresponding to the operation of certain sequential logic circuits. What type of logic relationships can sequential circuits model? In the following text, I will provide a simple example of this.

Jean Buridan's paradox sentence As an influential medieval French philosopher of his age, Jean (John) Buridan (c. 1295–1358) presented a puzzle with the following essence:

> Twelfth sophism: God exists and some conjunction is false.

John Buridan (2001: *Summulae de dialectica* (translated by Gyula Klima), Yale University Press, c.8, p.980)

Or in other words:

> God exists and none of the sentences in this pair is true.

What do you think about the truth value of these two sentences? Which of these two is true?

> p := God exists.
>
> q := Neither sentence p nor q is true.

'p' is true if God exists and false, otherwise. 'q' is true if neither p nor q is true.

Sentence q asserts a 'Not-OR' relation because 'neither p nor q' is equivalent to 'not (p or q)'. One component of the 'or' relation is an existential proposition, while the other is the 'or' relation itself. It is a peculiar sentence because it has a truth value, if it has any at all, which depends on itself. Therefore, it certainly cannot be translated into the classical first-order logic language.
Let us examine the logical possibilities. If p is true (i.e. God exists), then q is false because one of its components is true and the other is false. Consequently, the two together are false (i.e. q is false). The situation is not that simple if p is false (i.e. we deny God's existence). Suppose that q is true. This is possible only if both members are false. This is not, however, the case because the first member is false and the second member is true; hence, the result is false together and q cannot be