

# IMPRECISE PROBABILITY MODELS

## SPECIAL INSTANCES OF BELIEF STRUCTURES

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# INTRODUCTION

## WHAT IS THIS TALK ABOUT?

- The field of imprecise probabilities (IP) is very diverse, wide and complex.
- I will:
  - try to convey the essential ideas behind probabilistic reasoning;
  - show the similarities between reasoning in IP and in classical propositional logic;
  - use an unfamiliar approach and language.
- I will give a few examples to show that the theory:
  - is mathematically elegant;
  - is conceptually simple;
  - sometimes more so than precise probability.

# OUTLINE

- 1 SETS OF DESIRABLE GAMBLES
- 2 BELIEF STRUCTURES
- 3 DEALING WITH CONDITIONING

# SETS OF DESIRABLE GAMBLES

## BASIC NOTIONS AND NOTATIONS

- The possibility space  $\mathcal{X}$  is a set of elementary events  $x$ .
- We want to deal with uncertainty about which  $x$  in  $\mathcal{X}$  obtains.
- A gamble is a bounded real-valued map  $f: \mathcal{X} \rightarrow \mathbb{R}$ 
  - interpreted as an unknown reward;
  - expressed in units of some linear utility scale;
  - if  $x$  obtains, then the corresponding reward is  $f(x)$ .
- An event  $A$  is a subset of  $\mathcal{X}$ , and  $I_A$  its indicator (gamble).
- The set of all gambles on  $\mathcal{X}$  is denoted by  $\mathcal{L}$ .

# SETS OF DESIRABLE GAMBLES

## DESIRABILITY

- A subject finds a gamble desirable, when he accepts the following transaction:
  - we determine which  $x$  obtains;
  - the subject receives  $f(x)$  (positive or negative).
- A subject can collect gambles that are desirable to her into a set of desirable gambles  $\mathcal{D} \subseteq \mathcal{L}$ ;
- Such a set is considered to be a model for the subject's beliefs (behavioural dispositions).

# A DIGRESSION

STARTING OFF ON THE RIGHT FOOT



This is a personalist and behavioural account of epistemic probability, based on rationality principles that are considered to be normative.

The ideas behind this approach can be traced back to Bruno de Finetti, Peter Williams, and Peter Walley.

# SETS OF DESIRABLE GAMBLERS

## COHERENCE AS A RATIONALITY CRITERION

### DEFINITION (COHERENCE)

A set of desirable gambles  $\mathcal{D}$  is called **coherent** when for all gambles  $f, g$  on  $\mathcal{X}$ , and all real  $\lambda \geq 0$ :

D-1 if  $f < 0$  then  $f \notin \mathcal{D}$ ; [avoiding partial loss]

D-2 if  $f \geq 0$  then  $f \in \mathcal{D}$ ; [accepting partial gains]

D-3 if  $f \in \mathcal{D}$  and  $g \in \mathcal{D}$  then  $f + g \in \mathcal{D}$ ; [combination]

D-4 if  $f \in \mathcal{D}$  then  $\lambda f \in \mathcal{D}$ . [scaling]

We denote the set of all coherent sets of desirable gambles by  $\mathbb{D}$ :

$$\mathbb{D} := \{ \mathcal{D} \subseteq \mathcal{L} : \mathcal{D} \text{ is coherent} \}.$$

# SETS OF DESIRABLE GAMBLER

## COHERENCE IN GEOMETRICAL TERMS

A coherent set of desirable gambles is a **convex cone** receding from the origin, that includes the non-negative orthant, and has no point in common with the negative orthant.

# SETS OF DESIRABLE GAMBLER

## THE MEANING OF THE INCLUSION RELATION $\subseteq$

- Consider two sets of desirable gambles  $\mathcal{A}_1$  and  $\mathcal{A}_2$ .
- If  $\mathcal{A}_1 \subseteq \mathcal{A}_2$  then a subject with model  $\mathcal{A}_2$  accepts (at least) all the gambles that a subject with model  $\mathcal{A}_1$  accepts.
- So we can say that:
  - $\mathcal{A}_1$  is less committal than  $\mathcal{A}_2$ ;
  - $\mathcal{A}_1$  is more conservative than  $\mathcal{A}_2$ ;
  - $\mathcal{A}_1$  is less informative than  $\mathcal{A}_2$ .
- The most conservative, least committal, least informative coherent model is the non-negative orthant:

$$\mathcal{D}_+ := \{f \in \mathcal{L} : f \geq 0\}.$$

# SETS OF DESIRABLE GAMBLES

## DEFINITION OF LOWER AND UPPER PREVISIONS

- Consider a subject with a coherent set of desirable gambles  $\mathcal{D}$ .
- The lower prevision  $\underline{P}_{\mathcal{D}}(f)$  of any gamble  $f$  is then given by

$$\underline{P}_{\mathcal{D}}(f) := \sup \{ \mu \in \mathbb{R} : f - \mu \in \mathcal{D} \}$$

the subject's supremum acceptable buying price for  $f$ .

- The upper prevision  $\bar{P}_{\mathcal{D}}(f)$  of any gamble  $f$  is then given by

$$\bar{P}_{\mathcal{D}}(f) := \inf \{ \mu \in \mathbb{R} : \mu - f \in \mathcal{D} \}$$

the subject's infimum acceptable selling price for  $f$ .

- Lower and upper probabilities for an event  $A$ :

$$\underline{P}_{\mathcal{D}}(A) := \underline{P}_{\mathcal{D}}(I_A) \quad \text{and} \quad \bar{P}_{\mathcal{D}}(A) := \bar{P}_{\mathcal{D}}(I_A).$$

# SETS OF DESIRABLE GAMBLES

## PROPERTIES OF LOWER AND UPPER PREVISIONS

### THEOREM

*Let  $\mathcal{D}$  be a coherent set of desirable gambles, and let  $f, g$  be gambles on  $\mathcal{X}$ ,  $A$  a subset of  $\mathcal{X}$ , and  $\lambda \geq 0$  and  $\mu$  real numbers. Then:*

$$\text{LUPR-1 } \bar{P}_{\mathcal{D}}(f) = -\underline{P}_{\mathcal{D}}(-f) \text{ and } \bar{P}_{\mathcal{D}}(A) = 1 - \underline{P}_{\mathcal{D}}(A^c); [\text{conjugacy}]$$

$$\text{LUPR-2 } \inf f \leq \underline{P}_{\mathcal{D}}(f) \leq \bar{P}_{\mathcal{D}}(f) \leq \sup f; [\text{dominance}]$$

$$\text{LUPR-3 } \underline{P}_{\mathcal{D}}(f + g) \geq \underline{P}_{\mathcal{D}}(f) + \underline{P}_{\mathcal{D}}(g); [\text{super-additivity}]$$

$$\text{LUPR-4 } \underline{P}_{\mathcal{D}}(\lambda f) = \lambda \underline{P}_{\mathcal{D}}(f); [\text{non-negative homogeneity}]$$

$$\text{LUPR-5 } \underline{P}_{\mathcal{D}}(f + \mu) = \underline{P}_{\mathcal{D}}(f) + \mu; [\text{constant additivity}]$$

# SETS OF DESIRABLE GAMBLERS

WHAT ABOUT THE OTHER WAY AROUND?

- Coherent sets of desirable gambles are a more general and more expressive model than lower previsions:

*there is an infinity of coherent sets of desirable gambles  $\mathcal{D}$  that lead to the same lower prevision  $\underline{P}$ .*

- The ambiguity can be removed by imposing additional conditions on  $\mathcal{D}$ , such as:

D-5 if  $f + \varepsilon \in \mathcal{D}$  for all  $\varepsilon > 0$  then  $f \in \mathcal{D}$ . [closedness]

- In that case (uniquely):

$$\mathcal{D} = \{f \in \mathcal{L} : \underline{P}(f) \geq 0\}.$$

- We will incorporate D-5 from now on.

# SETS OF DESIRABLE GAMBLES

AND WHAT ABOUT PRECISE PROBABILITIES?

- When do we have a precise probability model, i.e.

$$\underline{P}_{\mathcal{D}}(f) = \overline{P}_{\mathcal{D}}(f) =: P(f) \text{ for all } f \in \mathcal{L}?$$

- If and only if in addition to D-1–D-5:  
D-6 if  $f \notin \mathcal{D}$  then  $-f \in \mathcal{D}$  for all  $f \in \mathcal{L}$ .
- In that case  $\mathcal{D} = \{f \in \mathcal{L} : P(f) \geq 0\} =: \mathcal{D}_P$  is a closed semi-space,
- and  $P$  is a coherent prevision in de Finetti's sense:  
PR-1  $\inf f \leq P(f)$ ; [normedness]  
PR-2  $P(f + g) = P(f) + P(g)$ ; [finite additivity]  
PR-3  $P(\lambda f) = \lambda P(f)$ . [homogeneity]

# SETS OF DESIRABLE GAMBLES

## PRECISE MODELS ARE MAXIMAL

- All the precise models  $\mathcal{D}_P$  are the maximal elements of the ordered set  $(\mathbb{D}, \subseteq)$ :

*they are not dominated by any other coherent model*

- Each coherent model  $\mathcal{D}$  is the intersection of all the maximal (precise) models  $\mathcal{D}_P$  that include it:

$$\mathcal{D} = \bigcap \{ \mathcal{D}_P : \mathcal{D} \subseteq \mathcal{D}_P \}$$

- Similarly for a lower prevision  $\underline{P}$ :

$$\mathcal{M}(\underline{P}) := \{ P : (\forall f \in \mathcal{L})(P(f) \geq \underline{P}(f)) \}$$

is a convex closed set of precise probability models, and

$$\underline{P}(f) = \min \{ P(f) : P \in \mathcal{M}(\underline{P}) \}.$$

# SETS OF DESIRABLE GAMBLER

## EXTENDING A SUBJECT'S ASSESSMENT

- Suppose a subject comes up with a set of gambles  $\mathcal{A}$  of gambles that she finds desirable: an assessment.
- We cannot expect her assessment  $\mathcal{A}$  to be already coherent, so generally  $\mathcal{A} \notin \mathbb{D}$ .
- We would like to extend  $\mathcal{A}$  to some coherent model  $\mathcal{D}$ , in a way that is as conservative as possible.

### DEFINITION (CONSISTENCY)

A set of desirable gambles is called consistent if it is included in some coherent set of desirable gambles:

$$(\exists \mathcal{D} \in \mathbb{D})(\mathcal{A} \subseteq \mathcal{D}).$$

# SETS OF DESIRABLE GAMBLES

## COHERENCE AS A BASIS FOR PROBABILISTIC REASONING

- Coherence is preserved by arbitrary intersections:

$$\mathcal{D}_i, i \in I \text{ coherent} \Rightarrow \bigcap_{i \in I} \mathcal{D}_i \text{ coherent.}$$

- This leads to the definition of a closure operator  $\text{cl}$ :

$$\text{cl}(\mathcal{A}) := \bigcap \{ \mathcal{D} \in \mathbb{D} : \mathcal{A} \subseteq \mathcal{D} \}.$$

### THEOREM (NATURAL EXTENSION)

- (I)  $\mathcal{A}$  is consistent iff  $\text{cl}(\mathcal{A}) \neq \mathcal{L}$ ;
- (II) if  $\mathcal{A}$  is consistent then its natural extension  $\mathcal{E}_{\mathcal{A}} := \text{cl}(\mathcal{A})$  is the smallest coherent set of desirable gambles that includes  $\mathcal{A}$ .

# A DIGRESSION

## ON CLOSURE AND CLOSURE OPERATORS

The operator  $\text{cl}$  has the following properties:

CL-1  $\text{cl}(\emptyset) = \mathcal{D}_+$  and  $\text{cl}(\mathcal{L}) = \mathcal{L}$ ;

CL-2  $\mathcal{A} \subseteq \text{cl}(\mathcal{A})$ ;

CL-3 if  $\mathcal{A}_1 \subseteq \mathcal{A}_2$  then  $\text{cl}(\mathcal{A}_1) \subseteq \text{cl}(\mathcal{A}_2)$ ;

CL-4  $\text{cl}(\text{cl}(\mathcal{A})) = \text{cl}(\mathcal{A})$ ;

CL-5  $\mathcal{A} = \text{cl}(\mathcal{A})$  iff  $\mathcal{A}$  is coherent or  $\mathcal{A} = \mathcal{L}$ .

but  $\text{cl}$  is not a topological closure because generally:

$$\text{cl}(\mathcal{A}_1 \cup \mathcal{A}_2) \neq \text{cl}(\mathcal{A}_1) \cup \text{cl}(\mathcal{A}_2).$$

# SETS OF DESIRABLE GAMBLES

LET US SUMMARISE

imprecise probability

---

set of desirable gambles

set inclusion  $\subseteq$

coherent

– preserved by intersections  $\cap$

consistent

precise probability model

propositional logic

---

set of propositions

set inclusion  $\subseteq$

deductively closed

– preserved by intersections  $\cap$

consistent

complete set

belief structures

---

belief state

partial order  $\leq$

coherent

– preserved by infima  $\inf$

consistent

maximal element

# BELIEF STRUCTURES

## DEFINITION OF A BELIEF STRUCTURE

### DEFINITION (BELIEF STRUCTURES)

A belief structure is a tuple  $(\mathbf{S}, \mathbf{C}, \leq, 1_{\mathbf{S}}, 0_{\mathbf{S}})$  such that

B-1  $(\mathbf{S}, \leq)$  is a complete lattice with bottom  $0_{\mathbf{S}}$  and top  $1_{\mathbf{S}}$ ;

B-2  $\mathbf{C}$  is a subset of  $\mathbf{S}$  that is closed under arbitrary infima, and such that  $1_{\mathbf{S}} \notin \mathbf{C}$ .

An element  $s$  of  $\mathbf{S}$  is called a belief state. An element  $c$  of  $\mathbf{C}$  is called coherent.

- The partial order  $\leq$ : is at most as informative as.
- We can define a closure operator  $\text{cl}: \mathbf{S} \rightarrow \mathbf{S}$  by:

$$\text{cl}(s) := \inf \{c \in \mathbf{C} : s \leq c\}.$$

and then a belief state  $s$  is called consistent if  $\text{cl}(s) \neq 1_{\mathbf{S}}$ .

# BELIEF STRUCTURES

## DEFINITION OF A STRONG BELIEF STRUCTURE

- Consider the set  $\mathbf{M}$  of maximal elements  $\mathbf{m}$  of  $\mathbf{C}$ :

$$(\forall \mathbf{c} \in \mathbf{C})(\mathbf{m} \leq \mathbf{c} \Rightarrow \mathbf{m} = \mathbf{c}).$$

### DEFINITION (STRONG BELIEF STRUCTURES)

A belief structure is called **strong** if it is dually atomic, i.e., if for all  $\mathbf{c} \in \mathbf{C}$ :

$$\mathbf{c} = \inf \{ \mathbf{m} \in \mathbf{M} : \mathbf{c} \leq \mathbf{m} \}.$$

- Observe that in that case:

$$\text{cl}(\mathbf{s}) = \inf \{ \mathbf{m} \in \mathbf{M} : \mathbf{s} \leq \mathbf{m} \}.$$

# BELIEF STRUCTURES

## EXAMPLES

- Possibility measures constitute a belief structure that is not strong.
- Classical propositional logic (CPL) leads to a strong belief structure, whose maximal elements are the complete theories (ultrafilters, possible worlds).
- Imprecise probability (IP) leads to a strong belief structure, whose maximal elements are the precise probability models.
- CPL can be embedded in IP, but not in precise probability!

# BELIEF STRUCTURES

## BELIEF CHANGE WITH BELIEF STRUCTURES

- We can give an account of AGM-like belief expansion and revision in any belief structure:  
*expand/revise coherent belief state with other belief state*
- Nearly all results and characterisations for AGM's special case of CPL carry over: the essential arguments are order-theoretic.
- Maximal elements generalise Grove's and Lewis's spheres.
- Only 'problem': contraction no longer satisfies the Harper and Levi identities.
- In the case of IP, we have expansion/revision that is not necessarily conditioning.

# CONDITIONING

## DEFINITION AND IMMEDIATE PROPERTIES

- Consider a coherent set  $\mathcal{D}$  of desirable gambles (D-1–D-4).
- We can define conditional lower/upper previsions:

$$\underline{P}_{\mathcal{D}}(f|B) := \sup \{ \mu \in \mathbb{R} : I_B(f - \mu) \in \mathcal{D} \}$$

$$\bar{P}_{\mathcal{D}}(f|B) := \inf \{ \mu \in \mathbb{R} : I_B(\mu - f) \in \mathcal{D} \}$$

- If  $\mathcal{D} = \mathcal{D}_P$  then  $P_{\mathcal{D}}(f|B)P(B) = P(fI_B)$ : Bayes's rule, but no problems if  $P(B) = 0$ .
- More generally:

$$\underline{P}_{\mathcal{D}}(I_B[f - \underline{P}_{\mathcal{D}}(f|B)]) = 0 \quad \text{Generalised Bayes Rule.}$$

# CONDITIONING

## THE DEFINITION OF CONGLOMERABILITY

### DEFINITION (CONGLOMERABILITY)

A set of desirable gambles  $\mathcal{D}$  is **conglomerable** with respect to a partition  $\mathcal{B}$  of  $\mathcal{X}$  if for all gambles  $f$  on  $\mathcal{X}$ :

$$\mathcal{D}\text{-}\mathcal{B} \quad (\forall B \in \mathcal{B})(I_B f \in \mathcal{D}) \Rightarrow f \in \mathcal{D}.$$

- Some people argue for conglomerability (Walley, Seidenfeld), others against it (de Finetti, Goldstein).
- If conglomerability is required, the approach described above can still be used: leads to a theory of coherent conditioning and natural extension that is slightly different from Walley's.

# A DIGRESSION

## WHY REQUIRE CONGLOMERABILITY?

- Consider an integer random variable (different from zero):

$$\mathcal{X} = \{+, -\} \times \mathbb{N}.$$

- If it is positive, it is sampled from the  $\sigma$ -additive distribution  $P(\cdot|+)$  with mass function:

$$p(n) = (1 - \theta)^{n-1} \theta, \quad n \in \mathbb{N}.$$

- If it is negative, it is sampled from a uniform distribution on  $\mathbb{N}$ :

$$P(A|-) = 0, \quad A \text{ finite.}$$

- It is positive or negative with equal probability:

$$P(f) = \frac{1}{2}P(f|+) + \frac{1}{2}P(f|-).$$

# A DIGRESSION

## ON FINITE ADDITIVITY

- $P(\cdot| -)$  cannot be  $\sigma$ -additive: its mass function is identically zero.
- There are (infinitely many) finitely additive  $P(\cdot| -)$  by the Hahn–Banach Theorem.
- These  $P(\cdot| -)$  are intangibles, i.e., inconstructible.
- Their lower envelope is a lower prevision that is constructible:

$$\underline{P}(f) = \sup_{A \text{ finite}} \inf_{n \in A^c} f(n).$$

# A DIGRESSION

## INCURRING A SURE LOSS

$$P(\{+\}) = \frac{1}{2} \quad P(\{+\}|n) = 1, \quad n \in \mathbb{N}$$
$$P(\{(+, n)\}) = \frac{1}{2}\theta(1 - \theta)^{n-1}$$
$$P(\{(+, n), (-, n)\}) = \frac{1}{2}\theta(1 - \theta)^{n-1}$$

incurs a sure loss, and is not compatible with a conglomerable coherent  $\mathcal{D}$ .