Probabilistic Interpretation of Independence Friendly Logics

Ondrej Majer

Institute of Philosophy
Academy of Sciences of the Czech Republic

(joint work with Tero Tulenheimo, University of Helsinki)

Probabilistic Interpretation of IF Logics

Game-theoretical semantics

Independence Friendly logics

Extension of IF for finite models

Extension of IF for infinite models

Conclusions

Game-theoretical semantics

Development

- started in Lorenzen's work in 1950'
- developed in 1980's by Jaakko Hintikka and Gabriel Sandu
- boom in 1990's - game semantics for a number of non-classical logics (modal logics, linear logic, ...)

Game-theoretical semantics

- two players: Eloise and Abelard
- truth of a (classical) formula φ in a model $\mathbf{M}$ and evaluation v
- zero-sum game with perfect information
- two roles: Proponent $\mathcal{V}$ and Opponent $\mathcal{F}$ (Eloise starts as $\mathcal{V}$)
- position of game: (sub)formula of φ and evaluation v
- each round of the game consists of a move of one of the players

Evaluation games for classical logic

Choice rules:

- $\varphi = \psi_1 \vee \psi_2$: $\mathcal{V}$ chooses whether to play (ψ_1, v) or (ψ_2, v)
- $\varphi = \psi_1 \wedge \psi_2$: $\mathcal{F}$ chooses whether to play (ψ_1, v) or (ψ_2, v)

Assignment rules:

- $\varphi = (\forall x)\psi$: $\mathcal{F}$ chooses $a \in M$, game continues as $(\psi, v[x : a])$
- $\varphi = (\exists x)\psi$: $\mathcal{V}$ chooses $a \in M$, game continues as $(\psi, v[x : a])$

Negation:

- $\varphi = \neg\psi$: $\mathcal{V}$ and $\mathcal{F}$ *switch the roles*, the game continues as (ψ, v)

Terminating rule:

- φ is an atomic formula: (the current) $\mathcal{V}$ wins iff $\mathbf{M} \models \varphi[v]$

Game theory - solutions

Strategy - a function, that for each history where a given player is to move returns a move to continue with.

Winning strategy - a strategy that leads to a winning position of the player no matter what the opponent does.

$\#(s, s') = 1$ for any strategy s'

Game theory - solutions

Strategy - a function, that for each history where a given player is to move returns a move to continue with.

Winning strategy - a strategy that leads to a winning position of the player no matter what the opponent does.

$\#(s, s') = 1$ for any strategy s'

Theorem(Zermelo) In a zero-sum game of two players of finite depth with perfect information one of the players has a winning strategy.

Game theory - solutions

Strategy - a function, that for each admissible position of a given player returns a move to continue with.

Winning strategy - a strategy that leads to a winning position of the player no matter what the opponent does.

$\#(s, s') = 1$ for any strategy s'

Theorem(Zermelo) In a zero-sum game of two players of finite depth with perfect information one of the players has a winning strategy.

Correspondence theorem $\mathcal{V}$ has a winning strategy for the $\mathbf{M}$-Game (φ, v) iff $\mathbf{M} \models \varphi[v]$

More game theory - imperfect information

A player at a given position might not be informed about previous move(s) of the other player, she cannot discriminate between several positions in a game tree.

She has to decide about the next move without the full knowledge of the history of the game.

More game theory - imperfect information

A player at a given position might not be informed about previous move(s) of the other player, she cannot discriminate between several positions in a game tree.

She has to decide about the next move without the full knowledge of the history of the game.

Uniform Strategy - a function, that for each *set* of indiscernible histories, where given player is to move returns a move to continue with.

More game theory - imperfect information

A player at a given position might not be informed about previous move(s) of the other player, she cannot discriminate between several positions in a game tree.

She has to decide about the next move without the full knowledge of the history of the game.

Uniform Strategy - a function, that for each *set* of indiscernible histories, where given player is to move returns a move to continue with.

The notion of winning strategy has to be changed - it is required to be *uniform* with respect to the positions indiscernible for the player.

More game theory - imperfect information

Question:
What is the result of introducing imperfect information in the logical games?

More game theory - imperfect information

Question:
What is the result of introducing imperfect information in the logical games?

It allows us to make a distinction between syntactical subordination ('syntactic scope') and logical dependence ('semantic scope').

$\forall x \exists y P(x, y)$

More game theory - imperfect information

Question:
What is the result of introducing imperfect information in the logical games?

It allows us to make a distinction between syntactical subordination ('syntactic scope') and logical dependence ('semantic scope').

$\forall x \exists y P(x, y)$

$\forall x \exists y_{/x} P(x, y)$

Independence Friendly logics

Syntax

- standard connectives: $\varphi \wedge \psi, \varphi \vee \psi, \neg\varphi, \forall\varphi(x), \exists\varphi(x)$,
- slashed connectives $\forall x_1 ... \forall x_n \exists y_{/x_1,...,x_n} \varphi(x_1, ..., x_n, y)$

Game Semantics

- standard connectives follow the standard rules
- slashed connective - the existential move with *imperfect information*
- $\forall x_1 ... \forall x_n \exists y_{/x_1,...,x_n} \varphi(x_1, ..., x_n, y)$: $\mathcal{V}$ chooses $a \in M$ ("not knowing the values of $x_1, ..., x_n$,"), the game continues as $(\psi, v[y : a])$

Truth of a formula

- a formula is true iff there is a *uniform* winning strategy for Eloise
- a formula is false iff there is a *uniform* winning strategy for Abelard
- a formula is *undetermined* otherwise

Independence Friendly logics

Undetermined formulas

Let the domain M be a set of natural numbers

- $\forall x \exists y_{/x} x = y$

Independence Friendly logics

Undetermined formulas

Let the domain M be a set of natural numbers

- $\forall x \exists y_{/x} x = y$
- $\forall x \exists y_{/x} x \neq y$

Independence Friendly logics

Undetermined formulas

Let the domain M be a set of natural numbers

- $\forall x \exists y_{/x} x = y$
- $\forall x \exists y_{/x} x \neq y$
- $\forall x \exists y_{/x} Even(x + y)$

Independence Friendly logics

Undetermined formulas

Let the domain M be a set of natural numbers

- $\forall x \exists y_{/x} x = y$
- $\forall x \exists y_{/x} x \neq y$
- $\forall x \exists y_{/x} Even(x + y)$
- $\forall x \exists y_{/x} Div(x + y, 3)$

Independence Friendly logics

Undetermined formulas

Let the domain M be a set of natural numbers

- $\forall x \exists y_{/x} x = y$
- $\forall x \exists y_{/x} x \neq y$
- $\forall x \exists y_{/x} Even(x + y)$
- $\forall x \exists y_{/x} Div(x + y, 3)$

In an actual play of the corresponding evaluation games the chances of players might substantially differ.

Independence Friendly logics

Undetermined formulas

Let the domain M be a set of natural numbers

- $\forall x \exists y_{/x} x = y$
- $\forall x \exists y_{/x} x \neq y$
- $\forall x \exists y_{/x} Even(x + y)$
- $\forall x \exists y_{/x} Div(x + y, 3)$

In an actual play of the corresponding evaluation games the chances of players might substantially differ.

A more fine-grained analysis of the undefined formulas is needed. It is quite natural to extend the set of truth values to $[0, 1]$ in the way that the value reflects the chances of the players.

Game theory - solutions

Nash equilibrium in a game of two players E, A is a pair of strategies $< s^*, t^* >$ such that none of the players can profitably deviate from them.

$\#(s^*, t^*) \geq \#(s, t^*)$ for any strategy s of E

$\#(s^*, t^*) \leq \#(s^*, t)$ for any strategy t of A

Game theory - solutions

Properties of Nash equilibria of strictly competitive (zero-sum) games

Let's have a (strategic) game with the actions $a_1, ...a_m$, $b_1, ...b_n$ and utility functions u_1, u_2 for the A, E respectively. Then a pair of strategies $< s, s' >, s \in \{a_i\}, s' \in \{b_j\}$ is a Nash equilibrium iff

$$max_i min_j u_1(a_i, b_j) = min_j max_i u_2(a_i, b_j)$$

Game theory - solutions

Properties of Nash equilibria of strictly competitive (zero-sum) games

Let's have a (strategic) game with the actions $a_1, ...a_m$, $b_1, ...b_n$ and utility functions u_1, u_2 for the A, E respectively. Then a pair of strategies $< s, s' >, s \in \{a_i\}, s' \in \{b_j\}$ is a Nash equilibrium iff

$$max_i min_j u_1(a_i, b_j) = min_j max_i u_2(a_i, b_j)$$

- NE can be obtained by maxminimazing or minmaximizing the utility function
- all NE yield the same payoffs,
- NE pairs are mutually interchangeable
- NE might not exist in general.

More game theory - mixed strategies

In a situation of uncertainty a player might use a random device to pick up a particular move.

Mixed strategy is a probability distribution $\sigma = \{p_i\}$, over the set of pure strategies (actions) $\{a_i\}$.

More game theory - mixed strategies

In a situation of uncertainty a player might use a random device to pick up a particular move.

Mixed strategy is a probability distribution $\sigma = \{p_i\}$, over the set of pure strategies (actions) $\{a_i\}$.

Mixed payoffs are defined as expected values of the lotteries given by the corresponding probability distributions

$\#(\sigma, \gamma) = \sum p_i \cdot q_j \cdot u(a_i, b_j)$

Mixed strategies

Mixed strategy Nash equilibrium is defined in the same way as the pure one

$\#(\sigma^*, \gamma^*) \geq \#(\sigma^*, \gamma)$ for any *mixed* strategy γ of A

$\#(\sigma^*, t^*) \geq \#(\sigma, \gamma*)$ for any *mixed* strategy σ of E

Mixed strategies

Mixed strategy Nash equilibrium is defined in the same way as the pure one

$\#(\sigma^*, \gamma^*) \geq \#(\sigma^*, \gamma)$ for any *mixed* strategy γ of A

$\#(\sigma^*, t^*) \geq \#(\sigma, \gamma*)$ for any *mixed* strategy σ of E

Theorem Every *finite* strategic game has a Nash equilibrium in *mixed strategies*.

Extension of IF - finite case

Let us have an IF language L and a model **M** with finite domain M. We define the value of a formula φ as *the equilibrium payoff* for Eloise in the corresponding evaluation game

- for a *finite domain* the statements above guarantee that the value always exists and is unique.

- for the classical fragment of L we get classical logic

- the extension is conservative - a projection to 3 values (everything strictly between 0 and 1 to Undefined) yields the standard IF values

Extension of IF - finite case

Explicit definition of connectives

$$|\varphi \wedge \psi| = min(|\varphi|, |\psi|)$$

Extension of IF - finite case

Explicit definition of connectives

$|\varphi \wedge \psi| = min(|\varphi|, |\psi|)$

$|\varphi \vee \psi| = max(|\varphi|, |\psi|)$

Extension of IF - finite case

Explicit definition of connectives

$$|\varphi \wedge \psi| = min(|\varphi|, |\psi|)$$

$$|\varphi \vee \psi| = max(|\varphi|, |\psi|)$$

$$|\neg\varphi| = 1 - |\varphi|$$

Extension of IF - finite case

Explicit definition of connectives

$$|\varphi \wedge \psi| = min(|\varphi|, |\psi|)$$

$$|\varphi \vee \psi| = max(|\varphi|, |\psi|)$$

$$|\neg\varphi| = 1 - |\varphi|$$

$$|\forall x \varphi(x)| = min_{a \in M}(|\varphi(x)|_{x=a})$$

Extension of IF - finite case

Explicit definition of connectives

$$|\varphi \wedge \psi| = min(|\varphi|, |\psi|)$$

$$|\varphi \vee \psi| = max(|\varphi|, |\psi|)$$

$$|\neg\varphi| = 1 - |\varphi|$$

$$|\forall x \varphi(x)| = min_{a \in M}(|\varphi(x)|_{x=a})$$

$$|\exists x \varphi(x)| = max_{a \in M}(|\varphi(x)|_{x=a})$$

Extension of IF - finite case

Explicit definition of connectives

$|\varphi \wedge \psi| = min(|\varphi|, |\psi|)$

$|\varphi \vee \psi| = max(|\varphi|, |\psi|)$

$|\neg\varphi| = 1 - |\varphi|$

$|\forall x\varphi(x)| = min_{a\in M}(|\varphi(x)|_{x=a})$

$|\exists x\varphi(x)| = max_{a\in M}(|\varphi(x)|_{x=a})$

For classical connectives we get the weak fragment of Łukasiewicz logic

Extension of IF - finite case

Explicit definition of connectives

$|\varphi \wedge \psi| = min(|\varphi|, |\psi|)$

$|\varphi \vee \psi| = max(|\varphi|, |\psi|)$

$|\neg\varphi| = 1 - |\varphi|$

$|\forall x \varphi(x)| = min_{a \in M}(|\varphi(x)|_{x=a})$

$|\exists x \varphi(x)| = max_{a \in M}(|\varphi(x)|_{x=a})$

For classical connectives we get the weak fragment of Łukasiewicz logic

$|\exists y_{/x} \varphi(x, y)| = ??$

Extension of IF - finite case

The simplest mixed strategy is even distribution of probabilities among the actions, $\sigma = \{p_i\}, p_i = 1/n$

Examples

Let $M = \{1, ...n\}$.

$\varphi = \forall x \exists y_{/x} x = y, |\varphi| = 1/n$

$|\forall x \exists y_{/x} x \neq y| = (n-1)/n$

$\forall x \exists y_{/x} Even(x+y) = 1/2$ for n even, $\frac{(n+1)/2}{n}$

$\forall x \exists y_{/x} Div(x+y, 3) = 1/3$ for n divisible by 3, ...

Even distribution does not yield in general the equilibrium payoff.

Extension of IF - finite case

How do we calculate equilibria for the imperfect information moves?

Extension of IF - finite case

How do we calculate equilibria for the imperfect information moves?

Even distribution

Extension of IF - finite case

How do we calculate equilibria for the imperfect information moves?

Even distribution

Elimination of equivalent actions + even distribution?

Extension of IF - finite case

How do we calculate equilibria for the imperfect information moves?

Even distribution

Elimination of equivalent actions + even distribution?

Elimination of equivalent actions + elimination of dominated actions + even distribution?

Extension of IF - finite case

How do we calculate equilibria for the imperfect information moves?

Even distribution

Elimination of equivalent actions + even distribution?

Elimination of equivalent actions + elimination of dominated actions + even distribution?

Elimination in a general case: if there is a single strategy for each player left after elimination it is an equilibrium pair (some NA might have been eliminated).

Extension of IF - finite case

Questions

- How complex is calculating NE - is (Elimination of equivalent actions + Elimination of dominated actions + Even distribution) enough?

Extension of IF - finite case

Questions

- How complex is calculating NE - is (Elimination of equivalent actions + Elimination of dominated actions + Even distribution) enough?

- Which values do we obtain in a model of the size n?

Extension of IF - finite case

Questions

- How complex is calculating NE - is (Elimination of equivalent actions + Elimination of dominated actions + Even distribution) enough?

- Which values do we obtain in a model of the size n?

- Do we obtain intermediate values in "genuinely infinite" games?

Extension of IF - infinite case

Mixed strategies for infinite games

Even distribution

- single outcome has probability 0
- no σ additivity (just finite additivity)
- payoff cannot be calculated as a sum of singletons
- stronger notion of equilibrium (up to outcomes of measure zero)

Non-even distribution

- single outcome has a positive probability (infinite series)
- we keep sigma additivity
- weaker notion of equilibrium (up to ϵ, if any)

Extension of IF - infinite case

In general there are several possibilities of extending Nash equilibrium to a class of infinite games

Well behaved set of actions

If the set of actions is a convex compact subset of Euclidian space and payoff functions are continuous, the mixed equilibria exist. (Katukani Fixed-point theorem).

We do not know if the conditions are satisfied in general. The theorem guarantees the correctness of the extension for infinite models, but does not give any nice explicit characterization.

Extension of IF - infinite case

Equilibrium payoff as a limit

$\#(\sigma, \gamma) = lim_{n \to \infty} \#(\sigma_n, \gamma_n)$, σ_n, γ_n are equilibria for the size n

Does not work, consider the following formula φ:

$$\exists x \forall y \exists z \forall u \exists v_{/y} (y = u) \leftrightarrow (z = v) \wedge (u = z \to y = v) \wedge (v \neq x) \wedge (u \neq v)$$

φ is true in the models with the domain of which has odd size or is infinite.

$lim_{n \to \infty} \#(\sigma_n, \gamma_n)$ does not exist as
$\#(\sigma_n, \gamma_n)$ is 1 for odd n and 0 for n even

$\#(\sigma, \gamma) = 1$ as φ is true in standard IF

Extension of IF - infinite case

Equilibrium up to the strategies of zero measure

The sets of actions A, B such that $\#(s, s') < \#(s, a), \#(s, s') < \#(b, s'), a \in A, b \in B$ has measure 0

Conclusion

We introduced an extension of a fragment of IF logic introducing the notion of a mixed strategy in the game semantics and defining the value of a formula as the equilibrium payoff in the corresponding evaluation game. We showed that the definition is correct for finite models.

Conclusion

We introduced an extension of a particular IF logic introducing the notion of a mixed strategy in the game semantics and defining the value of a formula as the equilibrium payoff in the corresponding evaluation game. We showed that the definition is correct for finite models.

Things to be done:

- properties of NE in genuinely infinite models (do we obtain a value strictly between 0 and 1 in genuinely infinite games?)
- extend to another fragments of IF logics
- properties of the resulting logical systems (what kind of many valued logics we get)
- relevance for the logical approach to probability? (cf. van Benthem)