OPERATOR THEORY AND HYPERCOMPLEX ANALYSIS (S6)

DANIEL ALPAY (CHAPMAN), UWE KAEHLER (AVEIRO), IRENE SABADINI (MILAN)

Monday 14:00-16:00 (SIBSR7) chair: Uwe Kähler

14:00-14:25 Fabrizio Colombo
The fine structures on the S-spectrum

14:30-14:55 David Eelbode
The power of the outer exponential

15:00-15:25 Paula Cerejeiras
Fock spaces - a general framework

15:30-15:55 Arran Fernandez
Fractional d-bar derivatives and fractional polyanalyticity

Monday 16:30-19:00 (SIBSR7) chair: Irene Sabadini

16:30-16:55 Milton Ferreira
The Teodorescu and the Π operators in octonionic analysis and applications

17:00-17:25 Dmitrii Legatiuk
Discrete octonionic analysis: a non-associative play

17:30-17:55 Qinhai Huo
Octonionic Hilbert spaces and para-linear operators

18:00-18:25 Ming Jin
Weakly Slice Analysis on Non-Symmetric Domains in Several Quaternionic Variables

Tuesday 14:00-16:00 (SIBSR7) chair: Daniel Alpay

14:00-14:25 Raul Quiroga-Barranco
The geometry of slice regular Möbius transformations on the quaternionic unit ball

14:30-14:55 Irene Sabadini
On some classes of infinite order differential operators in hypercomplex analysis

15:00-15:25 Swanhild Bernstein
The q-Dirac operator on the quantum vector space

15:30-15:55 Martha Zimmermann
The q-deformed Dirac operator and the q-Hamiltonian

Tuesday 16:30-19:00 (SIBSR7) chair: Milton Ferreira

16:30-16:55 Cristina Diogo
Characterizing quaternionic numerical range through complex numerical range

17:00-17:25 Peter Schlosser
The H^∞-functional calculus for the quaternionic fine structure of the S-spectrum

17:30-17:55 Duvan Cardona
Control theory for the heat equation for non-local elliptic pseudo-differential operators on compact Lie groups

18:00-18:25 Uwe Kähler
Ternary Grassmann algebras and white noise space analysis

Wednesday 11:40-12:40 (SIBSR7) chair: Dmitrii Legatiuk

11:40-12:05 Elena Luna
Integral Theorems in the Theory of Bicomplex Holomorphic Functions and their relations with hyperbolic curves

12:10-12:35 Daniel Alpay
Scaled global operators and Fueter variables on non-zero scaled hypercomplex numbers

Abstracts.

Daniel Alpay, Chapman University

Scaled global operators and Fueter variables on non-zero scaled hypercomplex numbers

Abstract. We present a new family of Fueter-like variables associated to the global operator, and view this in a general setting which encompasses quaternions and split quaternions. Indefinite metric spaces (Pontryagin and Krein spaces) occur in a natural way.

The talk is based on joint works with Ilwoo Cho, Kamal Diki and Mihaela Vajiac.

References

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Versions of both papers are available on arxiv.

Swanhild Bernstein, TU Bergakademie Freiberg

The q-Dirac operator on the quantum vector space

Abstract. We consider q-commuting variables x_i, x_j such that $x_i x_j = q x_j x_i, i \neq j$. This is the standard setting in quantum calculus. Furthermore, we use the symmetric q-difference (which has some physical meaning [1]) and the multiplication from the right- and left-hand side because the structure is noncommutative. The $U(\mathfrak{o})$-invariant Laplacian [2,3] has the form

$$\Delta_q = q^{n-1}\partial_1^2 + q^{n-2}\partial_2^2 + \ldots + \partial_n^2.$$

We define the Dirac operator as the factorization of the Laplacian, i.e.

$$D_q = \sqrt{q}^{n-1}\partial_1 e_1 + \sqrt{q}^{n-2}\partial_2 e_2 + \ldots + \partial_n e_n.$$

We will discuss some properties of the Dirac operator and give examples of how to construct monogenic polynomials from harmonic polynomials in the case of $n = 2$.

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Duván Cardona, Ghent University

Control theory for the heat equation for non-local elliptic pseudo-differential operators on compact Lie groups

Abstract. In this talk we discuss our recent results about spectral inequalities for eigenvalues and their applications to control theory. In [1] We extend the estimates proved by Donnelly and Fefferman, and by Lebeau and Robbiano for sums of eigenfunctions of the Laplacian (on a

compact manifold) to estimates for sums of eigenfunctions of any positive and elliptic pseudo-differential operator of positive order on a compact Lie group. Our criteria are imposed in terms of the positivity of the corresponding matrix-valued symbol of the operator. As an application of these inequalities, we obtain the null-controllability for diffusion models for elliptic pseudo-differential operators on compact Lie groups. General results are also discussed on compact manifolds. Joint work with Michael Ruzhansky and Julio Delgado.

References

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Paula Cerejeiras, University of Aveiro, Portugal

Fock spaces - a general framework

Abstract. We consider pairs of weighted shift operators on a weighted ℓ^2 space with weights associated with an entire function. When considered as operators over a general Fock space the commutators of these pairs of weighted shift operators are diagonal operators. We establish a calculus for the algebra of these commutators. As examples, we present the general case of Gelfond-Leontiev derivatives. This construction allows us to establish a general framework, which goes beyond the classic.

This is a joint work with D. Alpay, U. Kähler, and T. Kling.

Fabrizio Colombo, Politecnico di Milano

The fine structures on the S-spectrum

Abstract. Holomorphic functions play a crucial role in operator theory and the Cauchy formula is a very important tool to define functions of operators. The Fueter-Sce-Qian extension theorem is a two steps procedure to extend holomorphic functions to the hyperholomorphic setting. The first step gives the class of slice hyperholomorphic functions; their Cauchy formula allows to define the so-called S-functional calculus for noncommuting operators based on the S-spectrum. In the second step this extension procedure generates monogenic functions; the related monogenic functional calculus, based on the monogenic spectrum, contains the Weyl functional calculus as a particular case. In this talk we show that the extension operator from slice hyperholomorphic functions to monogenic functions admits various possible factorizations that induce different function spaces. The integral representations in such spaces allows to define the associated functional calculi based on the S-spectrum. The function spaces and the associated functional calculi define the so called *fine structure of the spectral theories on the S-spectrum.* Among the possible fine structures there are the harmonic and poly-harmonic functions and the associated harmonic and poly-harmonic functional calculi. In this talk we present the state of the art of this new branch of operator theory based on the S-spectrum.

Cristina Diogo, ISCTE-IUL and

Characterizing quaternionic numerical range through complex numerical range

Abstract. The numerical range, defined as the image of the unit sphere under a certain quadratic form, reveals different geometric structures depending on whether the ground field is the complex numbers $\mathbb{C}$ or the skew field of Hamilton's quaternions $\mathbb{H}$. The complex numerical range has been extensively studied in the literature, contrary to its quaternionic counterpart. In this talk, we explore the coexistence of both notions, and we bring to the quaternionic setting some results from the complex setting. In fact, for the class of complex operators acting on a quaternionic Hilbert space, we characterize the quaternionic numerical range in terms of the complex one and two suitable real numbers. Moreover, we provide a characterization of the quaternionic numerical range for normal operators.

Joint work with Luís Carvalho and Sérgio Mendes.

David Eelbode, University of Antwerp

The power of the outer exponential

Abstract. One of the most important (geometrical) properties of a bivector B in a (Euclidean) Clifford algebra is the fact that it can be decomposed as a sum $B = b_1 + \ldots + b_k$ of simple and mutually orthogonal bivectors b_j (where k is at most half the dimension of the underlying space). This result goes back to the infamous Cartan-Dieudonné theorem and a conjecture made by Marcel Riesz, but recently regained interest in the work of De Keninck and Roelfs (see [1]). The first thing we will do in this talk is to explain how the so-called outer exponential (introduced by P. Lounesto in his seminal book) can give some insight into the nature of these decomposing bivectors b_j, which then appear naturally as images of certain trigonometric functions. If time permits, we will also explain why these observations matter and how the bivectors b_j can be used to construct spinors as clearly defined geometrical objects.

References

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This is joint work with S. De Keninck and M. Roelfs.

Arran Fernandez, Eastern Mediterranean University

Fractional d-bar derivatives and fractional polyanalyticity

Abstract. The d-bar derivative, and the closely associated Dirac operator, are differential operators of vital importance in hypercomplex analysis, ranging from complex to quaternionic to Clifford analysis. We consider different possible ways to define fractional powers of these operators, including fractional derivatives of both Riemann–Liouville and Caputo type, in order to obtain a richer and more general theory. A naturally arising question, once fractional d-bar derivatives have been defined, is what are the kernels of these operators, the functions whose fractional d-bar derivatives are zero? We answer this question by giving a complete characterisation of the so-called fractionally polyanalytic functions in $\mathbb{C}$.

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Milton Ferreira, Polytechnic of Leiria, Portugal

The Teodorescu and the Π *operators in octonionic analysis and applications*

Abstract. In the octonionic function theory, the non-associativity together with the non-commutativity play a key role. For instance, the Stokes formula contains an additional associator term originated by the non-associative property. To take the non-associativity into account, particular intrinsic weight factors are implemented in the definition of octonion-valued inner products to ensure the existence of a reproducing Bergman kernel. This Bergman projection plays a pivotal role in the L_2-space decomposition for octonion-valued functions. We will address this question. Furthermore, we study an octonionic Teodorescu transform and show how it is related to the unweighted version of the Bergman transform and establish some operator relations between these transformations. We apply two different versions of the Borel-Pompeiu formulae that naturally arise in the context of the non-associativity.

Finally, we use the octonionic Teodorescu transform to establish a suitable octonionic generalization of the Ahlfors-Beurling Π-operator. We prove an integral representation formula that presents a unified representation for the Π-operator arising in all hypercomplex function theories, and describe some of its mapping properties. Applications of the Π-operator associated with the octonionic Beltrami equation and the hyperbolic octonionic Dirac operator will be shown.

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Qinghai Huo, Hefei University of Technology

Octonionic Hilbert spaces and para-linear operators.

Abstract. The theory of octonionic Hilbert spaces was first introduced by Goldstine and Horwitz in 1964 but subsequently remained dormant. Recently, it has been demonstrated that one of the axioms within this theory is not independent of the others. This revelation has led to the introduction of a novel concept known as octonionic para-linearity, which has emerged as the central focus of octonionic functional analysis. By substituting linearity with paralineartiy, the definition of Hilbert spaces over complex numbers, quaternions and octonions can be unified. The Riesz representation theorem also applies to paralinear functionals naturally. Furthermore, we have introduced the octonionic version of dual operators based on the Riesz representation theorem. Unexpectedly, by establishing a new polar identity for paralinear operators, we find a new characterization of octonionic paralinear self-adjoint operators. The relationships between the kernel and range of a paralinear operator and its dual operator also undergo significant changes. A novel notion of fractional subspaces is introduced to refine the classical relationships.

Ming Jin, Macau University of Science and Technology

Weakly Slice Analysis on Non-Symmetric Domains in Several Quaternionic Variables

Abstract. In this talk, we focus on the algebraic structure of weakly slice regular functions on several quaternionic variables in the frame of weakly slice analysis based on slice topology. Slice analysis is a generalization of function theory of one complex variable(or several complex variables) in high-dimensional spaces. With its development, the theory is further divided into strongly slice analysis and weakly slice analysis. Weakly slice analysis was only proposed in the past three years and the introduction of a new topology(i.e. slice topology) brings a new perspective to slice analysis. For the algebraic structure in weakly slice analysis, we introduced a $*$-product that preserves the slice-regular property. For this purpose, we proposed precise definitions for open neighborhoods of a path and the holomorphism of stem functions.

Uwe Kähler

Ternary Grassmann algebras and white noise space analysis

Abstract. Classic supersymmetry is based on $\mathbb{Z}_2$-graded algebras, like Clifford and Grassmann algebras which still allows us to consider a Fock space of monogenic function and build most of the necessary ingredients for a theory of entire functions. But more general settings like quarks need a more general type of supersymmetry based on a $\mathbb{Z}_3$-grading (also called hypersymmetry). In this talk we present the groundwork for an Itô/Malliavin stochastic calculus and Hida's white noise analysis in the context of a supersymmetry with $\mathbb{Z}_3$-graded algebras. To this end, we establish a ternary Fock space and the corresponding strong algebra of stochastic distributions and present its application in the study of stochastic processes in this context.

Dmitrii Legatiuk, Universität Erfurt

Discrete octonionic analysis: a non-associative play

Abstract. In this talk, we discuss ideas on discretisation of the classical octonionic analysis. In particular, we present the ideas related to a direct discretisation of partial derivatives by help of finite differences. We discuss the influence of the non-associativity of octonionic multiplication and present an explicit derivation of the associator in the discrete setting. Further, we present basics of discrete octonionic function theory, including Borel-Pompeiu formulae and Cauchy transforms.
This is a joint work with Rolf Sören Kraußhar and Anastasiia Legatiuk.

M. Elena Luna-Elizarrarás, Holon Institute of Technology, Holon, Israel

Integral Theorems in the Theory of Bicomplex Holomorphic Functions and their relations with hyperbolic curves

Abstract. In this talk we will analyze some geometric aspects of the well known Integral Theorems for holomorphic bicomplex functions. In particular we will reveal some connections between hyperbolic curves and those (1-dimensional) curves over which the integrals of such theorems are evaluated.

Raul Quiroga-Barranco, Centro de Investigación en Matemáticas, Mexico

The geometry of slice regular Möbius transformations on the quaternionic unit ball

Abstract. Let us denote by $\mathbb{B}$ the quaternionic unit ball centered at the origin. It has been developed in [1,2] a rich theory of hyperholomorphic quaternionic valued functions on $\mathbb{B}$ based on the notion of slice regularity. This includes, among many other topics, the introduction of slice regular Möbius transformations, which constitute a family that will be denoted by $\mathcal{M}(\mathbb{B})$.

We will present the construction, based on the notion of slice regularity, of some geometric structures that can be associated to both $\mathbb{B}$ and $\mathcal{M}(\mathbb{B})$. Our development generalizes the well-known situation for the complex case and the unit disk, but has to overcome some obstructions inherent to the non-commutative quaternionic case. Our results can be found in [3,4].

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Irene Sabadini, Politecnico di Milano

On some classes of infinite order differential operators in hypercomplex analysis

Abstract. Infinite-order differential operators appear in different fields of mathematics and physics and in the past decade they turned out to be of fundamental importance in the study of the evolution of superoscillations In this talk we discuss we investigate the continuity of a class of infinite-order differential operators acting on spaces of two different classes of entire hyperholomorphic functions. In both cases, entire hyperholomorphic functions with exponential bounds play a crucial role in the continuity of infinite-order differential operators. This is particularly remarkable since the exponential function is not in the kernel of the Dirac operator, and so it does not belong to this class of hyperholomorphic functions, but it still plays an important role.

Peter Schlosser, Politecnico di Milano

The H^∞-functional calculus for the quaternionic fine structure of the S-spectrum

Abstract. For slice-hyperholomorphic functions f, the Cauchy-formula

$$f(T) := \frac{1}{2\pi} \int_{\partial(\mathbb{C}_J \cap U)} S_L^{-1}(s,T) ds_J f(s), \tag{1}$$

also called the S-functional calculus, is a way to define functions of operators in quaternionic Banach spaces. However, there are two restrictions:

i) f has to be slice-hyperholomorphic,
ii) f has to admit some decay at ∞ in order to ensure the convergence of the integral.

In this talk I will present methods how to relax both conditions.

i) By changing the integral kernel $S_L^{-1}(s,T)$, we are able additionally treat harmonic, polyharmonic and axially monogenic functions f in an integral of the form (1). This method results in the so-called fine structure of the S-spectrum.
ii) The decay at ∞ on the other hand can be relaxed, by using regularizer functions e. With them one can define the so called H^∞*-functional calculus*, i.e. $f(T)$ is defined as

$$f(T) := e(T)^{-1}(ef)(T),$$

where on the right hand side, $e(T)$ and $(ef)(T)$ are both defined via (1).

Martha Zimmermann, TU Bergakademie Freiberg

The q-deformed Dirac operator and the q-Hamiltonian

Abstract. We consider Clifford algebras in the context of Jackson calculus. We start with establishing the foundations for this q-deformed setting and introduce q-deformed partial derivatives $\partial_i^q = \frac{f(x_1,\ldots,qx_i,\ldots,x_n)-f(x_1,\ldots,x_n)}{(q-1)x_i}$. The q-partial derivatives allow us to then define required operators such as the q-Dirac operator $D_x^q = \sum_{i=1}^n e_i\partial_i^q$, q-Euler operator $E^q = \sum_{i=1}^n x_i\partial_i^q$ and q-Gamma operator $\Gamma^q = \sum_{i<j} e_ie_j(x_i\partial_j^q - x_j\partial_i^q)$ in our new setting. Further, we will regard some of the relations between those operators.
The Dirac and Euler operator are related to the definition of a Hamiltonian. We transfer this to the q-deformed setting to obtain a q-Hamiltonian. This will also lead us to q-deformed raising and lowering operators and their relations to each other. A central object related to the Hamiltonian is the exponential function. As there are several different possibilities for a q-deformed exponential, part of this talk will focus on the right choice of exponential function for the considered q-Hamiltonian. Further, we will regard some properties of this q-exponential function.