ORTHOGONAL POLYNOMIALS AND SPECIAL FUNCTIONS (S2)

JACOB S. CHRISTIANSEN (LUND) AND ANA F. LOUREIRO (KENT)

Monday 14:00-16:00 (SIBLT2) chair: Ana Loureiro

14:00-14:25 Arno Kuijlaars
Matrix valued orthogonal polynomials arising from periodic hexagon tilings
14:30-14:55 Thomas Bothner
Universality for random matrices with an edge spectrum singularity
15:00-15:25 Wolter Groenevelt
Orthogonal polynomials and stochastic duality
15:30-15:55 Margit Rösler
Limits of Bessel functions for root systems as the rank tends to infinity

Monday 16:30-19:00 (SIBLT2) chair: Jacob Christiansen

16:30-16:55 Thomas Wolfs
Approximation of Euler's constant using multiple orthogonal polynomials
17:00-17:25 Maxim Yattselev
On smooth perturbations of Chebyshëv polynomials and $\bar{\delta}$-Riemann-Hilbert method
17:30-17:55 Mateusz Piorkowski
Non-hermitian contour orthogonality with rational weights: Applications to random tiling models
18:00-18:25 Daniel Perales
Finite free probability and hypergeometric polynomials

Wednesday 11:40-12:40 (PSR2) chair: Jacob Christiansen

11:40-12:05 Benjamin Eichinger
Universality limits and homogeneous de Branges spaces
12:10-12:35 Brian Simanek
Orthogonal polynomials and mutually unbiased bases

Wednesday 14:00–15:00 (PSR2) chair: Ana Loureiro

14:00-14:25 Grzegorz Świderski
Asymptotic zeros' distribution of orthogonal polynomials with unbounded recurrence coefficients
14:30-14:55 Andy Hone
Elliptic orthogonal polynomials from solutions of the Toda and Volterra lattice

Thursday 14:00-16:00 (PSR2) chair: Jacob Christiansen

14:00-14:25 Peter Clarkson
Classical solutions of the fifth Painlevé equation
14:30-14:55 Ben Mitchell
Special function solutions of the fifth Painlevé equation
15:00-15:25 Kerstin Jordaan
Orthogonal polynomials and symmetric Freud weights
15:30-15:55 Kenta Miyahara
The sinh-Gordon reduction of the Painlevé III

Thursday 16:30–18:30 (PSR2) chair: Ana Loureiro

16:30-16:55	Walter Van Assche *Unique special solution for discrete Painlevé II*
17:00-17:25	Michael Voit *Freezing limits of Calogero–Moser–Sutherland particle models*
17:30-17:55	Open Problem Session

Abstracts.

Thomas Bothner, University of Bristol

Universality for random matrices with an edge spectrum singularity

Abstract. We study invariant random matrix ensembles defined on complex Hermitian matrices with a single root type singularity and one-cut regular density of states. Assuming that the singularity lies within the soft edge boundary layer we compute asymptotics of the model's generating functional by using Riemann-Hilbert problems for orthogonal polynomials and integrable operators. This extends an old result by Forrester and Witte and is based on ongoing joint work with Toby Shepherd (Bristol).

Peter Clarkson, University of Kent

Classical solutions of the fifth Painlevé equation

Abstract. In this talk I will discuss classical solutions of the fifth Painlevé equation ($\mathrm{P_V}$)

$$\begin{aligned}\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{(w-1)^2(\alpha w^2+\beta)}{z^2 w} \\ + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1},\end{aligned} \tag{$*$}$$

where α, β, γ and δ are constants.

The general solutions of the Painlevé equations are transcendental in the sense that they cannot be expressed in terms of known elementary functions. However, it is well known that all Painlevé equations except the first equation possess rational solutions, algebraic solutions and solutions expressed in terms of the classical special functions for special values of the parameters. These solutions of the Painlevé equations are often called "classical solutions" and frequently can be expressed in the form of determinants.

In the generic case of $\mathrm{P_V}$ ($*$) when $\delta \neq 0$, special function solutions are expressed in terms of Kummer functions and has rational solutions expressed in terms of Laguerre polynomials. In the case of $\mathrm{P_V}$ ($*$) when $\delta = 0$, which is known as deg-$\mathrm{P_V}$ and related to the third Painlevé equation, special function solutions are expressed in terms of Bessel functions and has algebraic solutions expressed in terms of Laguerre polynomials. I shall give some new representations of some of these classical solutions and discuss Bäcklund transformation for $\mathrm{P_V}$ ($*$).

Joint work with Clare Dunning and Ben Mitchell (University of Kent).

References

[1] P.A. Clarkson, Classical solutions of the degenerate fifth Painlevé equation, *J. Phys. A: Math. Theor.* **56**, (2023), 134002.
[2] P.A. Clarkson, C. Dunning, Rational solutions of the fifth Painlevé equation. Generalised Laguerre Polynomials, *Stud. Appl. Math.*, **152** (2024) 453–507.

Benjamin Eichinger, TU Wien

Universality limits and homogeneous de Branges spaces

Abstract. Homogeneous de Branges spaces are certain reproducing kernel Hilbert spaces which satisfy an additional rescaling property. We recently showed that the corresponding reproducing kernels appear as limit kernels for regularly varying universality limits. These rescaling limits include in particular bulk universality and hard edge universality limits. In general, the limit kernels form a three parameter family explicitly expressible in terms of confluent hypergeometric

functions. In this talk, we discuss structural properties of these spaces and explain why they naturally appear as limit kernels for universality limits.

The talk is based on a joint work with Milivoje Lukić and Harald Woracek

Wolter Groenevelt, TU Delft

Orthogonal polynomials and stochastic duality

Abstract. Duality is a very useful tool in the study of interacting particle processes which allows properties of one process to be studied using properties of the dual process. In this talk I explain how certain families of orthogonal polynomials, for example Krawtchouk polynomials, appear as duality function for certain inclusion and exclusion processes.

Andrew Hone, University of Kent

Elliptic orthogonal polynomials from solutions of the Toda and Volterra lattice

Abstract. We construct a family of elliptic orthogonal polynomials associated with solutions of the Toda lattice that are doubly periodic in time. These arise from a continued fraction expansion, due to van der Poorten, which yields a J-fraction for a family of functions of order 2 on an elliptic curve, and generates Hankel determinant solutions of the Somos-4 recurrence. The orthogonal polynomials may naturally be considered as elliptic analogues of Chebyshev polynomials, and in the real case correspond to polynomials constructed by Akhiezer. The relation with solutions of the Volterra lattice will also be mentioned.

References

[1]Hone, Continued fractions and Hankel determinants from hyperelliptic curves, *Communications on Pure and Applied Mathematics* **74**, (2021), 2249-2479.
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Kerstin Jordaan, University of South Africa

Orthogonal polynomials and symmetric Freud weights

Abstract. In this talk I will present some results on polynomials orthogonal with respect to exponential weights on the real line, in particular the symmetric Freud weight

$$\omega(x;t,\tau,\rho) = |x|^\rho \exp\left\{-x^6 + \tau\, x^4 + t\, x^2\right\}, \qquad x \in \mathbb{R}$$

with τ, t and $\rho > -1$ parameters. For certain specific values of the parameters, the associated moments can be written as finite partition sums of generalised hypergeometric functions. The case where $\rho = 0$ is of particular interest and has been studied in the context of Hermitian one-matrix models and random symmetric matrix ensembles with researchers in the 1990s observing "chaotic, pseudo-oscillatory" behaviour of the recurrence relation coefficients. More recently, this "chaotic phase" was described as a dispersive shockwave in a hydrodynamic chain. I will describe properties of the recurrence coefficients in the three-term recurrence relation associated with the polynomials orthogonal with respect to this weight. Collaborators: Peter Clarkson and Ana Loureiro (University of Kent, UK).

Arno Kuijlaars, KU Leuven

Matrix valued orthogonal polynomials arising from periodic hexagon tilings

Abstract. Periodically weighted random tiling models may be analyzed with the help of matrix valued orthogonal polynomials (MVOP) [1]. I will consider a class of MVOP that arises from periodic lozenge tilings of a hexagon with period three and discuss their asymptotic properties as their degrees tend to infinity. A major role is played by an equilibrium measure on a three sheeted Riemann surface.

References

[1] M. Duits, A. B. J. Kuijlaars, The two-periodic Aztec diamond and matrix valued orthogonal polynomials, *J. Eur. Math. Soc.* **23**, (2021), 1075–1131.

Ben Mitchell, University of Kent

Special Function Solutions of the Fifth Painlevé Equation

Abstract. We explore rational solutions of the fifth Painlevé equation. This equation exhibits special function solutions for specific parameter values, expressed in terms of Kummer functions. It is well-known that the third Painlevé equation has special function solutions represented by Bessel functions. By utilizing connection formulae between Kummer functions and modified Bessel functions, we demonstrate that the fifth Painlevé equation also admits Bessel function solutions. Furthermore, we investigate the structure of the roots of these solutions.

References

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[2] T. Masuda, Classical Transcendental Solutions of the Painlevé Equations and Their Degeneration, Tohoku Math. J., 2004, 56, **4**, 467–490.
[3] P.J. Forrester and N.S. Witte, Application of the τ-function theory of Painlevé equations to random matrices: $\mathrm{P_V}$, $\mathrm{P_{III}}$, the LUE, JUE, and CUE, *Comm. Pure Appl. Math.*, **55** (2002), no. 6, 679–727.

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Kenta Miyahara, Indiana University Indianapolis

The sinh-Gordon reduction of the Painlevé III

Abstract. We consider the large x asymptotics of any real-valued solutions of the sinh-Godron reduction of the Painlevé III on the real line. One is the one-parameter family of solutions that are smooth near ∞; the other is the two-parameter family of solutions that have accumulated singularities near ∞. We apply the Riemann-Hilbert nonlinear steepest descent method of Deift and Zhou to the sinh-Gordon Painlevé III in view of its Lax integrability and obtain the desired asymptotic formulae. This is joint work with Alexander Its and Maxim Yattselev.

Daniel Perales, Texas A&M University

Finite free probability and hypergeometric polynomials

Abstract. If we consider a generalized hypergeometric series where a parameter on top is a negative integer, then we naturally obtain a polynomial. When we have few paramaters, these hypergeometric polynomials are well known orthogonal families (Laguerre and Jacobi) and in particular are real-rooted for some regions of parameters.

The finite free additive and multiplicative convolutions are binary operations of polynomials that behave well with respect to real roots. We will apply these convolutions to the basic families of polynomials to systematically construct more involved hypergeometric polynomials (with several parameters) that have all real roots.

Furthermore, since the polynomial convolutions can be understood as a finite analogue of free probability that involves only discrete measures, then we can automatically understand the asymptotic root distribution of hypergeometric polynomials as free multiplicative convolutions of the corresponding limiting measures. Our results can be applied to study multiple orthogonal polynomials.

References

[1] Andrei Martínez-Finkelshtein, Rafael Morales, and Daniel Perales. Real roots of hypergeometric polynomials via finite free convolution. International Mathematics Research Notices, 06 2024 (arXiv:2309.10970).
[2] Andrei Martínez-Finkelshtein, Rafael Morales, and Daniel Perales. Zeros of generalized hypergeometric polynomials via finite free convolution. Applications to multiple orthogonality, 2024 (arXiv:2404.11479).

Margit Rösler, Paderborn University

Limits of Bessel functions for root systems as the rank tends to infinity

Abstract. The asymptotic analysis of multivariate special functions has a long tradition in infinite dimensional harmonic analysis, tracing back to classical work of Olshanski, Vershik, and Kerov. Nowadays, such issues have become of renewed interest in the context of random matrix theory.

In this talk, we consider the asymptotic behaviour of Dunkl-type Bessel functions associated with root systems of type A and type B with positive multiplicities as the rank tends to infinity. In both cases, we characterize the possible limit functions and the sequences of spectral parameters for which such limits exist. In the type A case, we present an alternative and natural approach to recent results by Assiotis and Najnudel related to beta ensembles in random matrix theory. These results generalize classical facts about the approximation of the Olshanski spherical functions on the space of infinite-dimensional Hermitian matrices over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (with the action of the infinite unitary group) by spherical functions of finite-dimensional Hermitian matrix spaces. Our approach naturally carries over to the B-case, where it gives asymptotic results for the spherical functions associated with the Cartan motion groups of non-compact Grassmannians as special cases.

The talk is based on joint work with Dominik Brennecken (Paderborn).

References

[1] D. Brennecken, M. Rösler, Limits of Bessel functions for root systems as the rank tends to infinity. To appear in *Indag. Math.* (volume in honor of G. van Dijk);

https://doi.org/10.1016/j.indag.2024.05.004.

Brian Simanek, Baylor University

Orthogonal polynomials and mutually unbiased bases

Abstract. Two orthonormal bases of a finite dimensional vector space are called mutually unbiased if the Fourier coefficients of any basis vector in terms of the other basis all have the same magnitude. It is known that an n-dimensional space contains at most $n+1$ mutually unbiased bases and that this bound is sharp. It is unknown if the bound can be saturated for every n. This talk will present an approach to this problem using orthogonal polynomials on the unit circle and a negative result about the prospects of solving the problem with this method.

This is based on joint work with Graeme Reinhart (Baylor).

Grzegorz Świderski, Polish Academy of Sciences

Asymptotic zeros' distribution of orthogonal polynomials with unbounded recurrence coefficients

Abstract. We study spectrum of finite truncations of unbounded Jacobi matrices with periodically modulated entries. In particular, we show that under some hypotheses a sequence of properly normalized eigenvalue counting measures converge vaguely to an explicit infinite Radon measure. Finally, we derive strong asymptotics of the associated orthogonal polynomials in the complex plane, which allows us to prove that Cauchy transforms of the normalized eigenvalue counting measures converge pointwise and which leads to a stronger notion of convergence. This is a joint work with Bartosz Trojan (Wrocław University of Science and Technology).

Walter Van Assche, KU Leuven, Belgium

Unique special solution for discrete Painlevé II

Abstract. We show that the discrete Painlevé II equation with starting value $a_{-1} = -1$ has a unique solution for which $-1 < a_n < 1$ for every $n \geq 0$. This solution corresponds to the Verblunsky coefficients of a family of orthogonal polynomials on the unit circle. The proof uses an idea of Tomas Lasic Latimer [2] who used orthogonal polynomials on the real line. We also give an upper bound for this special solution.

References

[1] W. Van Assche, Unique special solution for discrete Painlevé II, *J. Difference Equations Appl.* **30** (2024), no. 4, 465–474.
[2] T. Lasic Latimer, Unique positive solutions to q-discrete equations associated with orthogonal polynomials, *J. Difference Equations Appl.* **27** (2021), no. 5, 763–775.

Michael Voit, Technische Universität Dortmund

Freezing limits of Calogero–Moser–Sutherland particle models

Abstract. One-dimensional Calogero–Moser–Sutherland particle models with N particles can be regarded as diffusions on Weyl chambers or alcoves in $\mathbb{R}^N$ with second order differential operators as generators, which are singular on the boundaries of the state spaces. The most relevant examples are multivariate Bessel and Heckman–Opdam processes which are related to special functions associated with root systems. These models include Dyson's Brownian motions, multivariate Laguerre and Jacobi processes and, for fixed time, β-Hermite, Laguerre,

and Jacobi ensembles. In some cases, they are related to Brownian motions on the classical symmetric spaces.

We review some freezing limits for fixed N when some parameter, an inverse temperature, tends to ∞. The limits are normal distributions and, in the process case, Gaussian processes. The parameters of the limits are described in terms of solutions of ordinary differential equations which are frozen versions of the particle diffusions. We discuss connections of these ODES with the zeros of the classical orthogonal polynomials and polynomial solutions of some one-dimensional inverse heat equations.

The talk is partially based on joint work with Sergio Andraus, Kilian Herrmann, and Jeannette Woerner.

References

[1] M. Voit, Freezing limits for Calogero–Moser–Sutherland particle models, *Studies Appl. Math.* **151**, (2023), 1230–1281.
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Thomas Wolfs, KU Leuven

Approximation of Euler's constant using multiple orthogonal polynomials

Abstract. Ever since its first appearance, it has been unknown whether Euler's constant γ is irrational. A common strategy to (try to) prove irrationality of a given constant is by means of rational approximation: if one can construct good enough rational approximants to the constant, its irrationality follows. I will discuss a construction based on the multiple orthogonal polynomials associated with the exponential integral investigated in [3]. Although the approximants will not be good enough to prove that Euler's constant is irrational, we will be able to improve the quality of the approximants studied in [1] and [2]. Afterwards, I will show that other, but related, constants are more susceptible to rational approximation and that certain Bessel-like multiple orthogonal polynomials can be used to explicitly prove their irrationality.

This work is part of my PhD with Walter Van Assche.

References

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[2] T. Rivoal, *Rational approximations for values of derivatives of the gamma function*, Trans. Amer. Math. Soc. **361** (11), 6115–6149 (2009).

[3] W. Van Assche, T. Wolfs, *Multiple orthogonal polynomials associated with the exponential integral*, Stud. Appl. Math. **151** (2023), 411–449.

Maxim Yattselev, Indiana University Indianapolis

On smooth perturbations of Chebyshëv polynomials and $\bar{\delta}$-Riemann-Hilbert method

Abstract. Rates of converges of the recurrence coefficients of orthogonal polynomials $P_n(z)$ satisfying orthogonality relations

$$\int_{-1}^{1} x^l P_n(x) \frac{\rho(x)dx}{\sqrt{1-x^2}} = 0, \quad l \in \{0, \dots, n-1\},$$

where $\rho(x)$ is a positive m times continuously differentiable function on $[-1,1]$, $m \geq 3$, will be discussed.