FRACTIONAL CALCULUS OPERATORS AND THEIR APPLICATIONS (S19)

ARRAN FERNANDEZ (EASTERN MEDITERRANEAN UNIVERSITY), MILTON FERREIRA (POLYTECHNIC OF LEIRIA), MANUELA RODRIGUES (UNIVERSITY OF AVEIRO), NELSON VIEIRA (UNIVERSITY OF AVEIRO)

Tuesday 14:00-16:00 (SIBSR5) chair: Nelson Vieira

14:00-14:25 Jaan Janno
Inverse problems to determine sources of fractional diffusion equations in a form of separated variables
14:30-14:55 Hafiz Muhammad Fahad
Generalised operational calculus approach for fractional differential equations
15:00-15:25 Haniyyah Ul Irshad
Transmuted fractional operators with general analytic kernels
15:30-15:55 M. Manuela Rodrigues
Generalized fractional gradient

Thursday 14:00-16:00 (SIBSR4) chair: Milton Ferreira

14:00-14:25 Arran Fernandez
Extending operational calculus and Sonine kernels to higher dimensions
14:30-14:55 Sunday Simon Isah
Bivariate fractional calculus with general bivariate analytic kernels
15:00-15:25 Katarzyna Gorska
Operator solutions for fractional Fokker-Planck equation and diffusion-wave equation
15:30-15:55 Tobiasz Pietrzak
The wave aspect of the solution to the generalized telegraph equation

Thursday 16:30–18:30 (SIBSR4) chair: Arran Fernandez

16:30-16:55 Katarzyna Ryszewska
Harnack estimates for parabolic-type problems with time nonlocalities
17:00-17:25 Nelson Vieira
Dirac's method applied to the time-fractional telegraph equation

Abstracts.

Hafiz Muhammad Fahad, National University of Sciences and Technology, Islamabad, Pakistan

Generalised Operational Calculus Approach for Fractional Differential Equations

Abstract. Mikusiński's operational calculus is a method for interpreting and solving fractional differential equations, formally similar to Laplace transforms but more rigorously justified. This formalism was established for Riemann–Liouville and Caputo fractional calculi in the 1990s, and more recently for other types of fractional calculus. In this talk, we consider the operators of Riemann–Liouville and Caputo fractional differentiation of a function with respect to another function, and discover that the approach of Luchko can be followed, with small modifications, in the more general settings too. We establish all the function spaces, formalisms, and identities required to build the versions of Mikusiński's operational calculus which cover Riemann–Liouville and Caputo derivatives with respect to functions. The mathematical structure established here is used to solve fractional differential equations using Riemann–Liouville and Caputo derivatives with respect to functions, the solutions being written using multivariate Mittag-Leffler functions, in agreement with the results found in other recent work.

It is useful to understand how the various operators of fractional calculus relate to each other, especially relations between newly defined operators and classical well-studied ones. If time allows, we will also focus on an important type of such relationship, namely conjugation relations, also called transmutation relations. We define a general abstract setting in which such relations are relevant, and indicate how they can be used to prove many results easily in general settings such as fractional calculus with respect to functions and weighted fractional calculus.

Arran Fernandez, Eastern Mediterranean University

Extending operational calculus and Sonine kernels to higher dimensions

Abstract. Mikusiński's operational calculus is an algebraic method for interpreting integro-differential operators and solving equations that involve these operators. Starting from the 1990s, it was used to solve, for the first time, multi-term incommensurate linear fractional differential equations [1,2]. These solutions were constructed in the Dimovski space

$$C_\alpha = \left\{ f : (0,\infty) \to \mathbb{C} \,\middle|\, f(t) = t^p f_1(t),\, p > \alpha,\, f_1 \in C[0,\infty) \right\},$$

which has also been used recently as a setting for the theory of Sonine kernels [3,4].

In the current work, we discuss how the method of Mikusiński's operational calculus, as well as the concept and spaces of Sonine kernels, can be extended to higher dimensions. Instead of considering functions f of a single variable, we study functions of several variables, and solve some simple fractional PDEs for such functions. Part of this work is soon to appear in [5].

References

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[3] Y. Luchko, General fractional integrals and derivatives with the Sonine kernels, *Math.* **9**(6), (2021), 594.
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Katarzyna Górska, Institute of Nuclear Physics, Polish Academy of Sciences

Operator solutions for fractional Fokker-Planck equation and diffusion-wave equation

Abstract. The evolution operator method will be applied to solve the fractional Fokker-Planck equation and diffusion wave equation in a $(1+1)$-dimensional setting involving time derivatives which are smeared using a power-law memory function. Two types of evolution operators with kernels determined by functions proportional to the stable distribution will be presented. It will be demonstrated that distinguishing which equations govern the spreading particles evolution, if based solely on the mean square displacement knowledge, becomes challenging for longer periods of time. The examination of evolution operators' evolution and their self-reproducing properties will also be conducted.

References

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Haniyyah Ul Irshad, National University of Sciences and Technology

Transmuted Fractional Operators with General Analytic Kernels

Abstract. Many different types of fractional calculus are defined by various kernel functions, and one example of a very general class is fractional calculus with analytic kernels. Others arise from transmuting the usual fractional calculus with invertible linear operators, for example, composition and multiplication operators. In this talk, we combine these two ideas to create a new and very general model of fractional calculus using analytic kernels with transmutations. We prove fundamental theorems of calculus and other results on function spaces and compositions in the framework of these general operators. As special cases, we obtain left-sided and right-sided operators with analytic kernels on arbitrary intervals, as well as operators with analytic kernels with respect to functions, weighted operators with analytic kernels, and more. We also briefly discuss the utility of a generalised Laplace transform for solving fractional differential equations in the setting of generalised transmuted fractional operators with analytic kernels.

Sunday Simon Isah, Eastern Mediterranean University

Bivariate Fractional Calculus with General Bivariate Analytic Kernels

Abstract. We use a general bivariate analytic function with fractional power substitutions to define a bivariate fractional integral operator which can be expressed as a double infinite sum of classical Riemann—Liouville operators, using analyticity, which allows many interesting and useful fundamental properties. We further consider inversion properties of our proposed model, which in turn motivate the definition of a bivariate fractional derivative operator based on the same bivariate analytic kernels modified by fractional powers. We then prove the analogues of the fundamental theorems of calculus, Leibniz rule, and consider the functional maps and bounds, Laplace, Fourier, and Mellin transforms in this model of bivariate fractional calculus. As an application, we consider some illustrative examples which have already found applications in the literature using our new model.

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Jaan Janno, Tallinn University of Technology

Inverse problems to determine sources of fractional diffusion equations in a form of separated variables

Abstract. We consider inverse problems to determine source terms of the form

$$F(t,x) = g(t)f(x) \qquad (*)$$

of a time-fractional diffusion equation of the order $\alpha \in (0,1)$ in a bounded domain Ω. In a first class of problems a normal derivative of u over a portion of a boundary of Ω in a time interval $(T-\epsilon, T)$ is given. Uniqueness of f is proved. Under the additional condition that α is irrational, the uniqueness of full F of the form $(*)$ is shown. The proofs use a branch line of a kernel of a solution operator of a direct problem. In a second class of problems an instant over-determination condition for u at $t = T$ is given. Uniqueness of g is shown. The proof uses asymptotics at ∞ of Mittag-Leffler functions involved in the solution operator of the direct problem.

Tobiasz Pietrzak, Institute of Nuclear Physics Polish Academy of Sciences

The wave aspect of the solution to the generalized telegraph equation

Abstract. In order to overcome the problem of the long transmission time of information in the transatlantic telegraph cable, scientists such as Maxwell, Lord Kelvin, and Heaviside began working on a profound understanding of the physics of electrical impulse propagation in long cables. An important achievement from these efforts was the so-called telegraphers' equation introduced by Heaviside. The telegraph equation originally used for transmission line analysis, also has other no less important applications, such as the description of heat transport in liquid helium or the description of the so-called bioheat transfer occurring in biological tissues. The telegraph equation has also been generalized using the fractional derivative concept. An equation of this type (generalized version) has been used to describe anomalous diffusion and chemical reactions, as well as biological applications such as heat conduction in muscles and blood. During the presentation, the generalized telegraph equation with the power-law memory function, its solution, and the frequency shift effect for the obtained solution will be presented.

M. Manuela Rodrigues, Nelson Vieira, CIDMA & University of Aveiro

Generalized fractional gradient

Abstract. Motivated by the increasing practical applications in fractional calculus, we study the classical gradient method under the perspective of the ψ-Hilfer derivative. This allows us to cover in our study several definitions of fractional derivatives that are found in the literature. We develop an algorithm for the ψ-Hilfer fractional order gradient method using a series representation of the target function. Using benchmark functions, the numerical method obtained by truncating higher-order terms was tested and analyzed. Considering variable order differentiation and optimizing the step size, the ψ-Hilfer fractional gradient method shows better results in terms of speed and accuracy. Our results generalize previous works in the literature.

This is a joint work with M. Ferreira (Polytechnic University of Leiria & CIDMA) and N. Vieira (CIDMA & University of Aveiro).

References

[1] N. Vieira, M.M. Rodrigues, and M. Ferreira, Fractional gradient methods via ψ-Hilfer derivative, *Fractal and Fractional*, **7**-No.3, (2023), Article No. 275 (30pp.).

Katarzyna Ryszewska, Warsaw University of Technology

Harnack estimates for parabolic-type problems with time nonlocalities

Abstract. The theory of Harnack inequalities is a wide and important topic in the analysis of elliptic and parabolic equations. It provides several properties of weak solutions to respective problems, the most significant of which is the Hölder continuity. Thus, developing this theory for nonlocal problems is very desirable. In the talk I will describe the main ideas regarding the application of de Giorgi-Nash -Moser theory for nonlocal problems. Firstly, to give an intuition, I will present the results concerning the Harnack estimates for parabolic-type problem with time fractional derivative of order $\alpha \in (0,1)$. Then, I will comment on recent results for problems with more general time nonlocality.

This is a joint work with Prof. Rico Zacher and Adam Kubica.

Nelson Vieira, CIDMA & University of Aveiro

Dirac's method applied to the time-fractional telegraph equation

Abstract. The free Dirac equation arises from the factorization of the Klein-Gordon equation using matrix coefficients satisfying anticommutation relations. In this talk, we focus on the factorization of the multidimensional time-fractional telegraph equation, applying Dirac's factorization method.

Explicit representations for solutions in the Fourier domain will be presented in terms of bivariate Mittag-Leffler functions and some results will be presented regarding their asymptotic behaviour at the origin and infinity. To obtain explicit representations of our solutions in the space-time domain, new results were obtained involving the inverse Fourier transform, bivariate Mittag-Leffler functions and two-variable Fox-H functions. Finally, some graphical representations of our solutions in the Fourier domain will be presented.

This is a joint work with M. Ferreira (Polytechnic University of Leiria & CIDMA) and M.M. Rodrigues (CIDMA & University of Aveiro).