SPECTRAL PROBLEMS AND COMPUTATION (S16)

CHRISTIANE TRETTER (BERN), MARCO MARLETTA (CARDIFF)

Wednesday 11:40-12:40 (SIBLT3) chair: Marco Marletta

11:40-12:05 Matteo Capoferri
Curl and asymmetric pseudodifferential projections
12:10-12:35 Karl Michael Schmidt
On the continuum limit for discrete Dirac operators on square lattices

Wednesday 14:00-14:25 (SIBLT3) chair: Karl Michael Schmidt

14:00-14:25 Siavash Sadeghi
Wavenumber-explicit bounds for first kind integral equations in wave scattering
14:30-14:55 Anna Rozanova-Pierrat
Spectral stability under domain convergence in a class of non-Lipschitz uniform domains

Thursday 14:00-16:00 (SIBLT3) chair: Matteo Capoferri

14:00-14:25 Christiane Tretter
Spectral bounds for damped systems
14:30-14:55 Iveta Semorádová
$\mathcal{PT}$-symmetric oscillators with one-center point interactions
15:00-15:25 Marko Lindner
Localisation of pseudospectra on discrete groups
15:30-15:55 Christian Wyss
Computing the quadratic numerical range

Thursday 16:30-19:00 (SIBLT3) chair: Christian Wyss

16:30-16:55 Matt Colbrook
Barriers and Classifications of Robust Koopman Learning
17:00-17:25 Sugirtha Gayathri
A study on the exponential spectrum in Banach algebras
17:30-17:55 Lyonell Boulton
Spectral analysis of Dirac operators with a purely imaginary dislocation

Abstracts.

Lyonell Boulton, Heriot-Watt University

Spectral analysis of Dirac operators with a purely imaginary dislocation

Abstract. In this talk we present a complete spectral analysis of Dirac operators with non-Hermitian matrix potentials of the form $i\,\mathrm{sgn} + V$ where $V \in L^1$. For $V = 0$ we compute explicitly the matrix Green function. This allows us to determine the spectrum, which is purely essential, and its different types. It also allows us to find sharp enclosures for the pseudospectrum and its complement, in all parts of the complex plane. Notably, this includes the instability region, corresponding to the interior of the band that forms the numerical range. Then, with the help of a Birman-Schwinger principle, we establish in precise manner how the spectrum and pseudospectrum change when $V \neq 0$, assuming the hypotheses $\|V\|_{L^1} < 1$ or $V \in L^1 \cap L^p$ where $p > 1$. We show that the essential spectra remain unchanged and that the ε-pseudospectrum stays close to the instability region for small ε. We determine sharp asymptotics for the discrete spectrum, whenever V satisfies further conditions of decay at infinity. Finally, in one of our main findings, we give a complete description of the weakly-coupled model.

The research has been conducted jointly with Tho Nguyen Duc and David Krejčiřík.

Matteo Capoferri, Heriot-Watt University

Curl and asymmetric pseudodifferential projections

Abstract. In my talk I will present a new approach to the spectral theory of systems of PDEs on closed manifolds, developed in a series of recent papers by Dmitri Vassiliev (UCL) and myself, based on the use of pseudodifferential projections. After discussing the general theory, I will turn to the (non-elliptic) operator curl, and explain how our techniques offer a new pathway to the study of spectral asymmetry.

References

[1] M. Capoferri, D. Vassiliev, Invariant subspaces of elliptic systems I: pseudodifferential projections, *Journal of Functional Analysis* **282** 8 (2022), 109402.
[2] M. Capoferri, D. Vassiliev, Beyond the Hodge Theorem: curl and asymmetric pseudodifferential projections, preprint arXiv:2309.02015.

This is joint work with D. Vassilev (UCL) which was partially supported by EPSRC Fellowship EP/X01021X/1.

Matthew J. Colbrook, University of Cambridge

Barriers and Classifications of Robust Koopman Learning

Abstract. Many modern dynamical systems are too complicated to analyze directly or we do not have access to models, driving significant interest in learning methods. Koopman operators, though classical, have recently emerged as a dominant approach because they allow the study of nonlinear dynamics using linear techniques by solving an infinite-dimensional spectral problem. However, current algorithms face challenges such as lack of convergence (e.g., spectral pollution), hindering practical progress. In this talk, I will explore the fundamental question: *When can we robustly learn spectral properties of Koopman operators from trajectory data of dynamical systems, and when can we not?* Understanding these boundaries is crucial for analysis, applications, and designing algorithms. We establish a foundational approach combining computational analysis and ergodic theory, revealing the first fundamental barriers – universal for

any algorithm – associated with system geometry and complexity, regardless of data quality and quantity. For instance, we demonstrate well-behaved smooth systems on tori where non-trivial eigenfunctions of Koopman operators cannot be determined by any sequence of (even randomized) algorithms, even with unlimited training data. Additionally, we identify when learning is possible and introduce optimal algorithms with verification that overcome issues in standard methods. These results pave the way for a sharp classification theory of data-driven dynamical systems, including beyond Koopman operators, based on how many limits are needed to solve a problem (the SCI hierarchy). These limits also characterize all previous methods. The talk is based on joint work with Igor Mezić and Alexei Stepanenko.

References

[1] Colbrook, Matthew J., Igor Mezić, and Alexei Stepanenko. "Limits and Powers of Koopman Learning." *arXiv preprint arXiv:2407.06312* (2024).

[2] Colbrook, Matthew J., and Alex Townsend. "Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems." *Communications on Pure and Applied Mathematics* 77.1 (2024): 221-283.

[3] Colbrook, Matthew J., Catherine Drysdale, and Andrew Horning. "Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators." *arXiv preprint arXiv:2405.00782* (2024).

[4] Colbrook, Matthew J. "Computing spectral measures and spectral types." *Communications in Mathematical Physics* 384 (2021): 433-501.

[5] Colbrook, Matthew J. "The mpEDMD algorithm for data-driven computations of measure-preserving dynamical systems." *SIAM Journal on Numerical Analysis* 61.3 (2023): 1585-1608.

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Sugirtha Gayathri, Indian Institute of Technology Hyderabad

A study on the exponential spectrum in Banach algebras

Abstract. In this talk, we give an outline of the literature about the notion of generalized exponentials in a Banach algebra and the corresponding spectrum. Towards a question of commutativity of the exponential spectrum, we investigate an associated set.

Marko Lindner, Hamburg University of Technology

Localisation of pseudospectra on discrete groups

Abstract. Given a bounded and possibly non-normal band operator A on $\ell^2(G)$ with a discrete group G and an integer $n \in \mathbb{N}$, we show that every nonzero vector $x \in \ell^2(G)$ has a finite subvector $x_{n,k}$ of size n at some (typically unknown) position $k \in G$ such that

$$\frac{\|Ax_{n,k}\|}{\|x_{n,k}\|} \le \frac{\|Ax\|}{\|x\|} + \varepsilon_n,$$

where $x_{n,k}$ was extended to G by zero, $\varepsilon_n \sim 1/\sqrt{n}$ and we know the proportionality constant.

Evaluating this inequality for an x that (almost) minimises $\frac{\|Ax\|}{\|x\|}$, one can draw conclusions about the norm of the inverse, resolvent norms, and pseudospectra of A versus those of its restrictions to subspaces $X_{n,k}$ of vectors with n-sized support.

As a result, we cover the pseudospectrum of A by a union of pseudospectra of its restrictions to spaces $X_{n,k}$, proving the absence of (pseudo)spectrum outside that union.

We show how to improve the result to $\varepsilon_n \sim 1/n$ and we explain the connection of ε_n to Dirichlet eigenvalues of an associated graph Laplacian.

This is joint work with Simon Chandler-Wilde (Reading) and Christian Seifert (Hamburg).

Anna Rozanova-Pierrat, Supélec

Spectral stability under domain convergence in a class of non-Lipschitz uniform domains

Abstract. I will present recent results from [**1**] and [**2**] (see also [**3**] for the independant on the boundary measure trace theory), focusing on the stability questions under domain convergence. Firstly, I will introduce the functional framework of the trace operator allowing to work with boundaries as supports of the upper regular Borel measures. Hence, these supports can define non-Lipschitz, and possibly fractal/multi-fractal, boundaries. In this framework, I introduce generalized Dirichlet, Neumann, and Robin problems for Poisson-type equations, for which we proved the Mosco convergence of the associated energy functionals along sequences of suitably converging domains. Generally, the Mosco convergence does not imply the operator norm convergence of resolvents. I will present the sufficient conditions on the domain convergence which imply a stability result for weak solutions, the norm convergence of the associated resolvents, and the convergence of the corresponding eigenvalues and eigenfunctions.

In the end of my talk, if I have some time, I will finish with the existence of optimal shapes for the Robin boundary problems (not known before [**1**]) in the parametrized classes of admissible domains in the sense that they minimize the initially given energy functionals. The keys for this result are the uniform on the shape of the domains the Poincaré inequality and the compactness of the introduced parametrized classes of admissible domains.

References

[1] M. Hinz, A. Rozanova-Pierrat, A. Teplyaev, Boundary value problems on Non-Lipschitz uniform domains: Stability, Compactness and the Existence of optimal shapes. *Asymp. Anal.* **134**, (2023), 25–61.
[2] M. Hinz, A. Rozanova-Pierrat, A. Teplyaev, Non-Lipschitz uniform domain shape optimization in linear acoustics. *SIAM J. Control Optim.* **59**, (2021), 1007–1032.
[3] G. Claret, M. Hinz, A. Rozanova-Pierrat, A. Teplyaev, Layer potential operators for transmission problems on extension domains. Preprint, 2024 https://hal.science/hal-04505158

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Siavash Sadeghi, University of Reading

Wavenumber-explicit bounds for first kind integral equations in wave scattering

Abstract. There has been significant interest in the derivation of wavenumber-explicit bounds for the inverses of operators arising in the boundary integral equation formulation of time-harmonic scattering problems, when the scatterer is a bounded Lipschitz domain (see e.g. [1, 2]). In the first part of this talk, we will obtain such bounds for the interior Dirichlet to Neumann map. Building on results from [2], we prove that the norm of the inverse of the boundary single-layer potential operator grows at worst as a polynomial function of the wavenumber, provided that a set of positive wavenumbers of arbitrarily small Lebesgue measure is excluded. This result holds even in cases where the exterior of the obstacle is strongly trapping, and although the integral equation fails to be uniquely solvable at every Dirichlet eigenvalue of the domain.

References

[1] S. N. Chandler-Wilde, E. A. Spence, A. Gibbs, and V. P. Smyshlyaev. High-frequency bounds for the Helmholtz equation under parabolic trapping and applications in numerical analysis. SIAM Journal on Mathematical Analysis, 2020.
[2] D. Lafontaine, E. A. Spence, and J. Wunsch. For most frequencies, strong trapping has a weak effect in frequency-domain scattering. Communications on Pure and Applied Mathematics, 2021.

This is joint work with my supervisor Simon N. Chandler-Wilde.

Karl Michael Schmidt, Cardiff University

On the continuum limit for discrete Dirac operators on square lattices

Abstract. The talk discusses the continuum limit of discrete Dirac operators on the two-dimensional square lattice as the mesh size tends to zero. We use the most natural and simplest embedding of the discrete Hilbert space into the continuum Hilbert space, and the question arises naturally when discretising the Dirac operator in two-dimensional Euclidean space, e.g. for numerical analysis. The discrete Dirac operator converges to the continuum Dirac operator in the strong resolvent sense, but not in the norm resolvent sense. The latter result is closely related to the observation that the Liouville theorem does not hold in discrete complex analysis. These results extend to the three-dimensional Dirac operator. This is joint work with Tomio Umeda.

Reference

[1] K.M. Schmidt, T. Umeda, Continuum limits for discrete Dirac operators on 2D square lattices, *Analysis and Mathematical Physics* **13**, (2023), 46.

Iveta Semorádová, Cardiff University & Czech Technical university

$\mathcal{PT}$-symmetric oscillators with one-center point interactions

Abstract. We investigate the spectrum of Schrödinger operators with imaginary polynomial potentials in $L^2(R)$, perturbed with δ, or δ' interaction, centered at the origin

$$-\partial_x^2 + ix^{2k-1} + \alpha\delta, \quad -\partial_x^2 + ix^{2k-1} + \beta\delta', \tag{1}$$

where $\alpha \in R$, $\beta \in R$, $k \in N$.

It is well established that the spectrum of the unperturbed operators consists of countable many real, isolated and simple eigenvalues for $k \geq 2$, and it is empty for $k = 1$.

When $\alpha \neq 0$ or $\beta \neq 0$, for $k \geq 1$, we observe countable many non-real eigenvalues appearing in complex conjugate pairs, and at maximum finitely many real eigenvalues. The non-real eigenvalues asymptotically converge to the eigenvalues of the unperturbed problems defined on $L^2(R_+)$ and $L^2(R_-)$ with Dirichlet, resp. with Neumann boundary conditions for δ, resp. δ' interaction.

Moreover, for $\alpha \leq C_k < 0$, we show the existence of negative real eigenvalue, diverging to $-\infty$ as $\alpha \to -\infty$.

References

[1] J. Behrndt, I. Semorádová, P. Siegl, The imaginary Airy operator with one-center δ interaction, *to appear in Pure and Applied Functional Analysis*
[2] M. Marletta, I. Semorádová, $\mathcal{PT}$-symmetric oscillators with one-center point interactions

manuscript in preparation

Christiane Tretter, University of Bern

Spectral bounds for damped systems

Abstract. In this talk we present enclosures for the spectra of operators associated with second order Cauchy problems for the case of non-selfadjoint damping. These new results yield much better bounds than the numerical range for both uniformly accretive and sectorial damping, and even in the case of selfadjoint damping. Applications e.g. to wave equations illustrate the results.
(joint work with B. Jacob, C. Trunk and H. Vogt as well as with N. Hefti)

Christian Wyss, University of Wuppertal

Computing the quadratic numerical range

Abstract. We present a new algorithm for the computation of the quadratic numerical range of a matrix. So far the canonical approach has been random vector sampling. While the random vector method works well for matrices of small dimension, it fails to compute the full quadratic numerical range of moderately sized matrices already. This is due to a concentration phenomenon, which makes it increasingly unlikely for the randomly computed points to lie close to the boundary of the quadratic numerical range. In our new algorithm we overcome this difficulty by using a steepest ascent procedure to generate sequences of points that converge to the boundary. We combine this with additional techniques to handle non-convex parts of the boundary and to correctly fill the interior. In a side-by-side comparison we illustrate that the resulting algorithm performs significantly better than random vector sampling.