NUMERICAL RANGES (S15)

ILYA SPITKOVSKY (NYU, ABU DHABI), TIN-YAU TAM (NEVADA)

Tuesday 16:30-18:30 (SIBSR6) chair: Tin-Yau Tam

16:30-16:55 Chi-Kwong Li, College of William & Mary, USA
Linear preserver problems and generalized numerical radius

17:00-17:25 Muyan Jiang, University of California, Berkeley, USA
Unified approach to reciprocal matrices with Kippenhahn curves containing elliptical components

17:30-17:55 Anne Greenbaum, University of Washington, USA
On the approximate rank and numerical range of the resolvent

18:00-18:25 Brooke Randell, University of California, Santa Cruz, USA
Exploring the numerical range of block Toeplitz operators

Wednesday 11:40-12:40 (SIBSR6) chair: Rute Lemos

11:40-12:05 Thomas Schulte-Herbruggen, Technische Universitẗ München, Germany
Quantum systems theory meets numerical ranges: New observability results in terms of C-numerical ranges

12:10-12:35 Edward Poon, Embry-Riddle Aeronautical University, USA
The simultaneous zero inclusion property and Birkhoff-James orthogonality

Wednesday 14:00-15:00 (SIBSR6) chair: Rute Lemos

14:00-14:25 Takeaki Yamazaki, Toyo University, Japan
Generalizations of Aluthge transform, numerical ranges and spectral radii

14:30-14:55 Nicholas Young, Newcastle University, UK
Operators with numerical range in an elliptical region

Thursday 14:00-16:00 (SIBSR6) chair: Douglas Farenick

14:00-14:25 Hugo Woerdeman, Drexel University, USA
Partial isospectrality of a matrix pencil and circularity of the c-numerical range

14:30-14:55 Stephan Weis, Czech Technical University in Prague, Czech Republic
Restricting states to a matrix algebra is an open map

15:00-15:25 Rute Lemos, University of Aveiro, Portugal
Algebraic curves associated to centrosymmetric matrices

15:30-15:55 Zinaida Lykova, Newcastle University, UK
The bfd-norm on spaces of analytic functions and the numerical range

Thursday 16:30-18:30 (SIBSR6) chair: Ilya Spitkovsky

16:30-16:55 Douglas Farenick, University of Regina, Canada
Matrix ranges and the Halmos dilation theorem in several variable operator theory

17:00-17:25 Jens de Vries, University of Twente, Netherlands
A Drury-type bound for ρ-contractions

17:30-17:55 Damian Kolaczek, University of Agriculture in Krakow, Poland
Numerical ranges of antilinear operators

18:00-18:25 Helena Soares, ISCTE, Portugal
Quaternionic essential numerical range of complex operators

Friday 14:00-16:00 (SIBSR6) chair: Chi-Kwong Li

14:00-14:25	Petr Blaschke, Silesian University in Opava, Czech Republic *Co-oval description of the boundary curve of the numerical range of a finite matrix*
14:30-14:55	Piotr Pikul, Jagiellonian University in Kraków, Poland *On the joint numerical range of triples of* 4×4 *Hermitian matrices*
15:00-15:25	Tin-Yau Tam, University of Nevada, Reno, USA *Generalized numerical ranges in Lie framework*

Abstracts.

Petr Blaschke, Silesian university in Opava

Co-oval description of the boundary curve of the numerical range of a finite matrix

Abstract. It is known that the numerical range of a finite matrix is a convex bounded set, so its boundary is a simple closed curve. Numerical range also contains all the eigenvalues. For a 2 by 2 matrix, the numerical range is just an ellipse with foci exactly at the eigenvalues. Is there a nice description of the boundary curve of a numerical range also for 3 by 3 matrices, 4 by 4 matrices and beyond in terms of eigenvalues? We will give such a description which includes not only eigenvalues but also other interesting points, some of which behave like "anti-eigenvalues".

Jens de Vries, University of Twente

A Drury-type bound for ρ-contractions

Abstract. We discuss a new Drury-type bound for (Sz.-Nagy–Foiaş) ρ-contractions for $\rho \in [1,2]$. In particular, this result interpolates von Neumann's inequality and Drury's classical bound [1], where the latter states: If A is any square matrix whose numerical radius is at most 1, then

$$\|p(A)\| \leq 1 - |p(0)|^2 + \sqrt{(1-|p(0)|^2)^2 + |p(0)|^2}$$

for any polynomial p with $\sup_{|z|\leq 1} |p(z)| \leq 1$. This is joint work with Felix Schwenninger.

References

[1] S.W. Drury, Symbolic calculus of operators with unit numerical radius, *Linear Algebra and its Applications*, 428(8-9):2061–2069, 2008.

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Douglas Farenick, University of Regina

Matrix ranges and the Halmos dilation theorem in several variable operator theory

Abstract. The well-known theorem of P.R. Halmos [1] concerning the existence of unitary dilations for contractive linear operators acting on Hilbert space is extended to d-tuples of contractive Hilbert space operators satisfying a certain matrix-positivity condition. Such operator d-tuples are called, herein, Toeplitz-contractive, and a characterisation of the Toeplitz-contractivity condition is presented. The matrix-positivity condition leads to the definition of a new metric for operator tuples, and some of the properties of this metric are explored. Lastly, the elements of the closed unit ball in this metric are shown to coincide with the matrix ranges or operator ranges of $(U, U^2, \ldots, U^n)$, where U is a universal unitary operator.

References

[1] P.R. Halmos, Normal dilations and extensions of operators, *Summa Brasil. Math.* **2**, (1950), 125–134.

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Anne Greenbaum, University of Washington

On the approximate rank and numerical range of the resolvent

Abstract. Let A be a square matrix and let λ be a simple eigenvalue of A with unit right and left eigenvectors x and y: $Ax = \lambda x$, $y^*A = \lambda y^*$. If $z \in \mathbb{C}$ is *much* closer to λ than to any other eigenvalue of A, then the resolvent $(A - zI)^{-1}$ is approximately equal to the rank one matrix $(\lambda - z)^{-1}xy^*$. If λ is *ill-conditioned*, then y is almost orthogonal to x, and for z very close to λ, the numerical range of the resolvent is approximately equal to a disk of radius $\frac{1}{2}|(\lambda - z)^{-1}|$ about the point $\frac{1}{2}|(\lambda - z)^{-1}|\ |y^*x|$.

We describe conditions under which $(A - zI)^{-1}$ closely resembles a rank one matrix, with numerical range approximately equal to a disk about a point close to the origin, even when z is much further away from an ill-conditioned eigenvalue λ. In this case, $(A - zI)^{-1} \approx \sigma_1(z)u_1(z)v_1(z)^*$, where $\sigma_1(z)$ is the largest singular value of $(A - zI)^{-1}$ and $u_1(z)$ and $v_1(z)$ are the associated left and right singular vectors. To see this, we start with the matrix $A_0 = A - \lambda I$, which has a 0 singular value whose right and left singular vectors are the same as the corresponding eigenvectors. Assuming that these singular vectors are almost orthogonal to each other (i.e., that λ is an ill-conditioned eigenvalue) and that the second smallest singular value of A_0 is well-separated from 0, we show that $(A_0 - zI)^{-1}$ resembles such a rank one matrix for a wide range of z values.

References

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Muyan Jiang, University of California, Berkeley

Unified approach to reciprocal matrices with Kippenhahn curves containing elliptical components

Abstract. *Reciprocal matrices* are tridiagonal matrices $(a_{ij})_{i,j=1}^{n}$ with constant main diagonal and such that $a_{i,i+1}a_{i+1,i} = 1$ for $i = 1, \ldots, n-1$. For these matrices, criteria are established under which their Kippenhahn curves contain elliptical components or even consist completely of such. These criteria are in terms of system of homogeneous polynomial equations in variables $(|a_{j,j+1}| - |a_{j+1,j}|)^2$, and established via a unified approach across arbitrary dimensions. The results are illustrated, and specific numerical examples provided, for $n = 7$ thus generalizing earlier work in the lower dimensional setting.

Damian Kołaczek, University of Agriculture in Krakow

Numerical ranges of antilinear operators

Abstract. We study numerical ranges of antilinear operators acting on Hilbert and Banach spaces. We discuss various similarities and differences between numerical radii and numerical ranges in linear and antilinear setting. Our main result is proving that the numerical ranges of antilinear operators on at least two-dimensional space are always discs, which improve previously known results [1,2] stating that such numerical ranges on Hilbert and Banach spaces are always annuli. We also introduce the concept of an *antilinear numerical index* of the space and compare it with ordinary numerical index based on linear operators.

The talk is based on recent research with Vladimir Müller (Institute of Mathematics, Czech Academy of Sciences) [3].

References

[1] M. Chø, I. Hur, J.E. Lee, Numerical ranges of conjugations and antilinear operators on Banach spaces, *Filomat* **35**, (2021), 2715–2720.
[2] I. Hur, J.E. Lee, Numerical ranges of conjugations and antilinear operators, *Linear Multilinear A.* **69**, (2021), 2990–2997.
[3] D. Kołaczek, V. Müller, Numerical ranges of antilinear operators, *Integr. Equat. Oper. Th.* **96**, (2024), 17.

Rute Lemos, University of Aveiro

Algebraic curves associated to centrosymmetric matrices

Abstract. The boundary generating curves and numerical ranges of some centrosymmetric matrices of orders up to 6 are characterized in terms of the matrices entries [1]. These results extend previous ones concerning Kac-Sylvester matrices [2]. The classification of all the possible boundary generating curves for centrosymmetric matrices of higher dimensions remains open. Illustrative figures of the obtained results are presented. This talk is based on a joint work with Natália Bebiano and Graça Soares.

References

[1] N. Bebiano, R. Lemos and G. Soares, Algebraic curves associated with centrosymmetric matrices of orders up to 6. *Adv. Oper. Theory* **9** (2024), 56.
[2] N. Bebiano, R. Lemos and G. Soares, On the numerical range of Kac-Sylvester matrices, *Electron. J. Linear Algebra* **39** (2023), 241–259.

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Chi-Kwong Li, College of William and Mary

Linear preserver problems and generalized numerical radius

Abstract. We discuss linear preserver problems related to the generalized numerical range and generalized numerical radius. Recent results and open problems will be presented.

Zinaida Lykova, Newcastle University

The bfd-norm on spaces of analytic functions and the numerical range

Abstract. Let E be the open region in the complex plane bounded by an ellipse. The B. and F. Delyon norm $\|\cdot\|_{\mathrm{bfd}}$ on the space $\mathrm{Hol}(E)$ of holomorphic functions on E is defined by

$$\|f\|_{\mathrm{bfd}} \stackrel{\mathrm{def}}{=} \sup_{T \in \mathcal{F}_{\mathrm{bfd}}(E)} \|f(T)\|,$$

where $\mathcal{F}_{\mathrm{bfd}}(E)$ is the class of operators T such that the closure of the numerical range $W(T)$ of T is contained in E. The name of the norm recognizes a celebrated theorem of the brothers Delyon, which implies that $\|\cdot\|_{\mathrm{bfd}}$ is equivalent to the supremum norm $\|\cdot\|_\infty$ on $\mathrm{H}ol(E)$.

We show that there exists an interesting connection between the bfd norm on $\mathrm{Hol}(E)$ and the supremum norm $\|\cdot\|_\infty$ on the space $\mathrm{H}^\infty(G)$ of bounded holomorphic functions on the symmetrized bidisc, the domain G in $\mathbb{C}^2$ defined by

$$G \stackrel{def}{=} \{(z+w, zw) : |z| < 1, |w| < 1\}.$$

It transpires that there exists a holomorphic embedding $\tau : E \to G$ having the property that, for any bounded holomorphic function f on E,

$$\|f\|_{\mathrm{bfd}} = \inf\{\|F\|_\infty : F \in \mathrm{H}^\infty(G), F \circ \tau = f\},$$

and moreover, the infimum is attained at some $F \in \mathrm{H}^\infty(G)$.

We also consider connections between operators T with the closure of $W(T)$ contained in E and Douglas-Paulsen operators.

The talk is based on joint work with Jim Agler and Nicholas Young [1].

References

[1] J. Agler, Z. A. Lykova and N. J. Young, On the operators with numerical range in an ellipse, *J. Funct. Anal.*, **2**87(8) Article 110556 (2024) https://doi.org/10.1016/j.jfa.2024.110556.

Piotr Pikul, Jagiellonian University in Kraków

On the joint numerical range of triples of 4×4 *Hermitian matrices*

Abstract. Joint numerical ranges of hermitian matrices are of interest due to their relation to quantum states. The JNR of a triple of hermitian matrices of order 4 represents expected values of three observables applied to all possible two-qubit states.

$$W\big(A_1, A_2, A_3\big) = \Big\{\big(\operatorname{tr}(PA_j)\big)_{j=1}^3 : P \in M_4,\ P \geq 0,\ \operatorname{tr}(P) = 1\Big\}.$$

The configuration of flat portions of the boundary (faces) was studied in case of triples of hermitian 3×3 matrices [1]. As could be expected, in dimension 4 both shape of the faces and their configuration can be of higher complexity. In the talk there will be presented current results on such JNRs.

This is an ongoing project in collaboration with Konrad Szymański, Stephan Weis, Karol Życzkowski and Ilya Spitkowsky.

References

[1] K. Szymański, S. Weis, K. Życzkowski, Classification of joint numerical ranges of three hermitian matrices of size three, *Linear Alg. and Appl.*, **545**, (2018), 148–173.

[2] K. Szymański et al., What is the shape of states of two qubits? `https://quantumstat.es/note/Two-qubit-3D-JNR`

Edward Poon, Embry-Riddle Aeronautical University

The simultaneous zero inclusion property and Birkhoff-James orthogonality

Abstract. A normed space $\mathcal{X}$ is said to have the Simultaneous Zero Inclusion (S0I) property if, for every invertible bounded linear operator T on $\mathcal{X}$, 0 lies in the spatial numerical range of T if and only if 0 lies in the spatial numerical range of T^{-1}. Previously the only known spaces with the S0I property were inner product spaces, corresponding to the classical numerical range. By connecting the S0I property to Birkhoff-James orthogonality, we show that there are non-inner product spaces based on Radon planes that have the S0I property.

Brooke Randell, University of California, Santa Cruz

Exploring the numerical range of block Toeplitz operators

Abstract. We will discuss the numerical range of a family of Toeplitz operators with symbol function $\phi(z) = A_0 + zA_1$, where A_0 and A_1 are 2×2 matrices with complex-valued entries. A special case of a result proved by Bebiano and Spitkovsky in 2011 states that the closure of the numerical range of the Toeplitz operator $T_{\phi(z)}$ is the convex hull of $\{W(\phi(z)) : z \in \partial\mathbb{D}\}$. Here, $W(\phi(z))$ denotes the numerical range of $\phi(z)$. We combine this result with the envelope algorithm to describe the boundary of the convex hull of $\{W(\phi(z)) : z \in \partial\mathbb{D}\}$. We also place specific conditions on the matrices A_0 and A_1 so that $\{W(\phi(z)) : z \in \partial\mathbb{D}\}$ is a set of potentially degenerate circular disks. The convex hull of $\{W(\phi(z)) : z \in \partial\mathbb{D}\}$ takes on a wide variety of shapes, including the convex hull of limaçons.

Thomas Schulte-Herbrüggen, Technical University of Munich (TUM)

Quantum systems theory meets numerical ranges: New observability results in terms of C-numerical ranges

Abstract. By way of example we connect key-notions of quantum systems and control theory with numerical ranges.

The well-known C-numerical range of A defined as $W(C,A) := \{\mathsf{tr}(C^\dagger UAU^\dagger)\,|\,U \in \mathbf{U}(n)\}$ is the projection of the unitary orbit of A onto C under the Hilbert-Schmidt scalar product. In quantum control A can be seen as an initial state (the reachable set of which under Hamiltonian dynamics equals its unitary orbit) and C as a quantum observable. If system dynamics are limited to proper subgroup orbits $\mathcal{O}_{\mathbf{K}}(A) := \{KAK^\dagger\,|\,K \in \mathbf{K} \subsetneq \mathbf{U}(n)\}$, one arrives at the so-called restricted C-numerical range $W_{\mathbf{K}}(C,A)$ [1].

New observability results in quantum systems theory [2] are taken over to establish symmetry conditions for A and C in relation to $\mathbf{K}$ under which the restricted C-numerical range of A exhausts the full C-numerical range, i.e. $W_{\mathbf{K}}(C,A) = W(C,A)$. In turn, similar symmetry arguments classify tomographiable pairs (A,C) as well as scenarios when $W_{\mathbf{K}}(C,A)$ collapses to a singleton *beyond* scalar A or C—the latter case being necessary and sufficient for the full $W(C,A)$ to be a singleton [3,4,5].

These results are placed into a symmetry-based Lie-theoretical framework of quantum systems theory.

Relies on joint work with Markus Wiener.

References

[1] G. Dirr, U. Helmke, M. Kleinsteuber, S.J. Glaser, and T. Schulte-Herbrüggen, Relative C-Numerical Ranges for Applications in Quantum Control and Quantum Information, *Lin. Multilin. Alg.* **56**, (2008), 27—51.
[2] T. Schulte-Herbrüggen and M. Wiener, in preparation, (2024).
[3] M. Marcus and M. Sandy, Three Elementary Proofs of the Goldberg-Straus Theorem on Numerical Radii, *Lin. Multilin. Alg.* **11**, (1982), , 243–252.
[4] C.K. Li, The C-Convex Matrices, *Lin. Multilin. Alg.* **21**, (1987), 303–312.
[5] C.K. Li, C-Numerical Ranges and C-Numerical Radii, *Lin. Multilin. Alg.* **37**, (1994), 51–82.

Helena Soares, ISCTE

Quaternionic essential numerical range of complex operators

Abstract. We study the essential numerical range of complex operators on a quaternionic Hilbert space and its relation with the essential S-spectrum. We give a new characterization of the essential numerical range relating it to the complex essential numerical range. Moreover, we show that the quaternionic essential numerical range of a normal operator is the convex hull of the essential S-spectrum.

Tin-Yau Tam, University of Nevada, Reno

Generalized numerical ranges in Lie framework

Abstract. Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$, K connected subgroup of G with Lie algebra $\mathfrak{k}$, $B_\theta(\cdot,\cdot)$ the inner product on $\mathfrak{g}$ induced by the Killing form $B(\cdot,\cdot)$, where θ is the Cartan involution of $\mathfrak{g}$ associated with $\mathfrak{k}$. For $X, C \in \mathfrak{g}$, the C-numerical range of X is

$$W_C(X) := \{B_\theta(C, \mathrm{Ad}(k)X) : k \in K\}, \quad C, X \in \mathfrak{g}.$$

Geometric properties of $W_C(X)$ and related results will be discussed.

Stephan Weis, Czech Technical University in Prague

Restricting states to a matrix algebra is an open map

Abstract. Two topological problems appeared in matrix theory about ten years ago. One of them [1] concerns the map $f : x \mapsto \langle x|Ax\rangle$ from the unit sphere of $\mathbb{C}^n$ to the numerical range, where $A \in M_n$ is a complex $n \times n$-matrix. The other one [2] involves the linear map $g : D_n \to \mathbb{R}^k, \rho \mapsto (\mathrm{Tr}(\rho A_1), \dots, \mathrm{Tr}(\rho A_k))$ on the convex set of density matrices $D_n = \{\rho \in M_n : \mathrm{Tr}(\rho) = 1, \rho \text{ is psd}\}$, where $A_1, \dots, A_k \in M_n$ are hermitian matrices. It turns out [3] that the openness of f is equivalent to that of g in case $k = 2$ (taking $A = A_1 + \mathrm{i}A_2$).

The openness of g is relevant in physics, as it governs the continuity of the maximum-entropy inference map, and of other inference maps [2]. For example, the phenomenon that a smooth change of local observations can lead to a discontinuous change of global inference states is discussed as a signal of a quantum phase transition [4].

In the theory of operator systems [5], the openness of g is equivalent to the openness of the orthogonal projection (with respect to the Frobenius inner product) of D_n onto the operator system L defined as the complex span of the identity matrix and the matrices $A_1, \dots, A_k$. This, in turn, is equivalent to the openness of the linear map that restricts states on M_n to L (by virtue of the Riesz representation theorem).

This talk presents a novel theorem that asserts that the map restricting states to an operator system L is an open map if L is a *-algebra. This result extends a theorem by Vesterstrøm [6] into a non-commutative setting. It simplifies the topological analysis of the map that restricts states to an operator system included in a proper *-subalgebra.

References

[1] D. Corey, C. R. Johnson, R. Kirk, B. Lins, and I. Spitkovsky, *Continuity properties of vectors realizing points in the classical field of values*, Linear and Multilinear Algebra 61:10, 1329–1338 (2013).
[2] S. Weis, *Continuity of the maximum-entropy inference*, Communications in Mathematical Physics 330:3, 1263–1292 (2014).
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[5] D. R. Farenick, *Extremal matrix states on operator systems*, Journal of the London Mathematical Society 61:3, 885–892 (2000).
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Hugo J. Woerdeman, Drexel University

Partial isospectrality of a matrix pencil and circularity of the c-numerical range

Abstract. We study when functions of the eigenvalues of the pencil

$$\mathrm{Re}(e^{-it}A) = \cos(t)\mathrm{Re}A + \sin(t)\mathrm{Im}A \tag{1}$$

are constant functions of t. The results are then applied to questions regarding the numerical range, the higher rank numerical range and the c-numerical range, and we derive trace type conditions for when these numerical ranges are disks centered at 0. The theory of symmetric polynomials plays an important part in the proofs. This talk is based on joint work with Alma van der Merwe and Madelein Thiersen.

Takeaki Yamazaki, Toyo University

Generalizations of Aluthge transform, numerical ranges and spectral radii

Abstract. In this talk, we shall introduce two types of generalizations of the Aluthge transformation, the one is called the induced Aluthge transform and another is called the spherical Aluthge transform. In this talk, we shall introduce (i) inclusion relations of numerical ranges among induced Aluthge transforms and (ii) a characterization of the Taylor spectrum radius via spherical Aluthge transforms. This is partially joint work with Professor Kais Feki.

Nicholas Young, Newcastle University (emeritus)

Operators with numerical range in an elliptical region

Abstract. We give new necessary and sufficient conditions for the numerical range $W(T)$ of a bounded linear operator T on a Hilbert space $\mathcal{H}$ to be a subset of the closed elliptical set $K_\delta \subseteq \mathbb{C}$ given by

$$K_\delta \stackrel{\mathrm{def}}{=} \left\{ x + iy : \frac{x^2}{(1+\delta)^2} + \frac{y^2}{(1-\delta)^2} \leq 1 \right\},$$

where $0 < \delta < 1$. We start by generalizing Berger's well-known criterion for an operator to have numerical radius at most one, his so-called *strange dilation theorem.* Specifically, we show that, for $\delta \in (0,1)$ and for an operator $T \in \mathcal{B}(\mathcal{H})$, $W(T) \subseteq K_\delta$ if and only if there exist a Hilbert space $\mathcal{K}$, an isometry $I : \mathcal{H} \to \mathcal{K}$ and a unitary operator U on $\mathcal{K}$ such that

$$\frac{1 - \frac{1}{2}zT}{1 - zT + \delta z^2} = I^* \frac{1}{1 - zU} I$$

for all $z \in \mathbb{D}$.

We next generalize the lemma of Sarason that describes power dilations in terms of semi-invariant subspaces to operators T that satisfy $W(T) \subseteq K_\delta$. This generalization yields a characterization of the operators $T \in \mathcal{B}(\mathcal{H})$ such that $W(T)$ is contained in K_δ in terms of certain structured contractions that act on $\mathcal{H} \oplus \mathcal{H}$. As a corollary of our results we extend Ando's parametrization of operators having numerical range in a disc to those T such that $W(T) \subseteq K_\delta$. Indeed, if $\dim \mathcal{H} < \infty$, then $W(T) \subseteq K_\delta$ if and only if there exist contractions A, B on $\mathcal{H}$ with $A = A^*$ such that

$$T = 2\sqrt{\delta}A + (1-\delta)\sqrt{1+A}B\sqrt{1-A}.$$

The talk is based on joint work with Jim Agler and Zinaida Lykova.

References

[1] J. Agler, Z. A. Lykova and N. J. Young, On the operators with numerical range in an ellipse, *J. Funct. Anal.*, **287**(8) Article 110556 (2024) https://doi.org/10.1016/j.jfa.2024.110556.